MULTIPLIER IDEAL SHEAVES, JUMPING NUMBERS, AND THE RESTRICTION FORMULA

QI'AN GUAN AND XIANGYU ZHOU

Abstract. In the present article, we establish an equality condition in the restriction formula on jumping numbers by giving a sharp lower bound of the dimension of the support of a related coherent sheaf. As applications, we obtain equality conditions in the restriction formula on complex singularity exponents by giving the dimension, the regularity and the transversality of the support, and we also obtain some sharp equality conditions in the fundamental subadditivity property on complex singularity exponents. We also obtain two sharp relations on jumping numbers.

1. Backgrounds and Motivations

Multiplier ideal sheaves associated to plurisubhamonic functions and their associated invariants (say, complex singularity exponents i.e. log canonical threshold (lct) in algebraic geometry and jumping numbers) have become in recent years a fundamental tool in several complex variables and algebraic geometry, and have been developing with great success by many mathematicians (see e.g. [35, 41, 8, 5, 29, 42, 43]).

Various important and fundamental properties about the multiplier ideal sheaves and the invariants have been established, such as the first properties: e.g., coherence, integrally closedness, Nadel vanishing theorem; and further properties: e.g., the restriction formula and subadditivity property (see e.g. [13, 14, 9]). Very recently, strong openness property of the multiplier ideal sheaf is established by our solution of Demailly's strong openness conjecture (see e.g. [26]).

In the present article, we'll discuss the restriction formulas for multiplier ideal sheaves and on jumping numbers, and related subadditivity property. Based on the strong openness property and some other recent results, we establish sharp equality conditions in the restriction formula and subadditivity property in Demailly-Ein-Lazaseld's paper [13] and Demailly-Kollár's paper [14], by giving sharp lower bounds of the dimensions of the support of the related coherent analytic sheaves. We also discuss some new properties about the multiplier ideal sheaves.

1.1. Organization of the paper.

In the present section, we recall the backgrounds and the motivations of the problems about sharp equality conditions in the restriction formula on jumping

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numbers and the fundamental subadditivity property on complex singularity exponents (Problem 1.1, Problem 1.2 and Problem 1.3).

In Section 2, we present the main results of the present paper: the solution of Problem 1.1 (Theorem 2.1, main theorem), the solutions of Problem 1.2 and Problem 1.3 (Theorem 2.2, Theorem 2.3 and Theorem 2.4, applications of Theorem 2.1); two sharp relations on jumping numbers (Corollary 2.2 of Theorem 2.5 and Theorem 2.6) and the slicing result on complex singularity exponents (Remark 2.5). In Section 3, we recall or give some preliminary results used in the proof of the main theorem and applications. In Section 4, we prove the main theorem (Theorem 2.1). In Section 5, we prove the applications of the main theorem (Theorem 2.2, Theorem 2.3, Remark 2.2 and Proposition 2.1). In Section 6, we prove the two sharp relations on jumping numbers (Corollary 2.2 of Theorem 2.5 and Theorem 2.6). In section 7, we present a relationship between the fibrewise Bergman kernels and integrability.

1.2. **Restriction formula and subadditivity property.**

Let Ω be a domain in $\mathbb{C}^n$ with coordinates $(z_1, \cdots, z_n)$ and origin $o = (0, \cdots, 0) \in \Omega$. Let u be a plurisubharmonic function on Ω. Nadel [35] introduced the multiplier ideal sheaf $\mathcal{I}(u)$ which can be defined as the sheaf of germs of holomorphic functions f such that $|f|^2e^{-2u}$ is locally integrable. Here u is regarded as the weight of $\mathcal{I}(u)$.

It is well-known that the multiplier ideal sheaf $\mathcal{I}(u)$ is coherent and integral closed, satisfies Nadel's vanishing theorem [35] and the restriction formula and subadditivity property, and the strong openness property $\mathcal{I}(u) = \cup_{\varepsilon>0}\mathcal{I}((1+\varepsilon)u)$ [24, 25, 26] i.e. our solution of Demailly's strong openness conjecture (the background and motivation of the conjecture could be referred to [8, 9]).

Let $I \subseteq \mathcal{O}_o$ be a coherent ideal. The jumping number $c_o^I(u)$ is defined (see e.g. [30, 31])

$$c_o^I(u) := sup\{c \geq 0 : |I|^2 \exp{(-2cu)} \text{ is integrable near } o\},$$

which can be reformulated by $c_o^I(u) := sup\{c \geq 0 : \mathcal{I}(cu)_o \supseteq I\}$.

Especially, when $I = \mathcal{O}_o$, the jumping number is just the complex singularity exponent denoted by $c_o(u)$ (see [45], see also [8, 9]) (or log canonical threshold by algebraic geometer see [38, 33]).

By Berndtsson's solution ([3]) of the openness conjecture $\mathcal{I}(c_o(u)u)_o \neq \mathcal{O}_o$ posed in [14], it follows that $\{z|c_z(u) \leq c_o(u)\} = Supp(\mathcal{O}/\mathcal{I}(c_o(u)u))$, which is an analytic set since $\mathcal{I}(c_o(u)u)$ is a coherent ideal sheaf [35] and the support of a coherent analytic sheaf is analytic.

Let $\mathcal{F} \subseteq \mathcal{O}$ be a coherent ideal sheaf. By the definition of $c_z^{\mathcal{F}_z}(u)$ and the strong openness property, it follows that $c_z^{\mathcal{F}_z}(u) > p \Longrightarrow \mathcal{F}_z \subseteq \mathcal{I}(pu)_z$ and $c_z^{\mathcal{F}_z}(u) \leq p \Longrightarrow \mathcal{F}_z \not\subseteq \mathcal{I}(pu)_z$.

Combining the fact that the support of a coherent analytic sheaf is an analytic subset, one obtains

The lowerlevel set of jumping numbers $\{z|c_z^{\mathcal{F}_z}(u) \leq p\} = Supp(\mathcal{F}/(\mathcal{F} \cap \mathcal{I}(pu)))$ *is an analytic subset.*

Let $H = \{z_{k+1} = \cdots = z_n = 0\}$. In [13] (see also (14.1) in [9]), the following restriction formula for multiplier ideal sheaves has been stated by rephrasing Ohsawa-Takegoshi L^2 extension theorem:

Restriction formula (for multiplier ideal). $\mathcal{I}(u|_H) \subseteq \mathcal{I}(u)|_H$.

Using the strong openness property, it follows that the above restriction formula for multiplier ideal is equivalent to

Restriction formula (on jumping number). Let I be a coherent ideal on $\mathcal{O}_{o'}$, where o' is the origin in H. Then $c^I_{o'}(u|_H) \leq \sup\{c^{\tilde{I}}_o(u)|\tilde{I} \subseteq \mathcal{O}_o \ \&\ \tilde{I}|_H = I\}$, where o' emphasizes that $c^I_{o'}(u|_H)$ is computed on the submanifold H.

When $I = \mathcal{O}_{o'}$, the restriction formula about jumping numbers degenerates to the following restriction formula (an "important monotonicity result" as said in [14]) about complex singularity exponents:

Proposition 1.1. [14]$c_{o'}(u|_H) \leq c_o(u)$, *where* $u|_H \not\equiv -\infty$.

In [14] (see also (13.17) in [9]), the following fundamental subadditivity property of complex singularity exponents has been presented:

Theorem 1.1. [14] *Let I and J be coherent ideals on $\mathcal{O}_o$. Let $u = \log|I|$ and $v = \log|J|$. $c_o(\max\{u, v\}) \leq c_o(u) + c_o(v)$.*

Let $o_1 \in \Omega_1$ and $o_2 \in \Omega_2$, and let $\pi_i : \Omega_1 \times \Omega_2 \to \Omega_i$ be projections for $i \in \{1, 2\}$. Motivated by the proof of Theorem 1.1 in [14] (see also (13.17) in [9]) and using Theorem 3.4 in [25, 27] (i.e., our solution of a conjecture posed by Demailly and Kollar in [14]), we obtain

Proposition 1.2. *Let I_1 and I_2 be coherent ideals in $\mathcal{O}_{o_1}$ and $\mathcal{O}_{o_2}$ respectively, u and v be plurisubharmonic functions near o_1 and o_2 respectively, then one has*

$$c^{I_1\times I_2}_{(o_1,o_2)}(\max\{u \circ \pi_1, v \circ \pi_2\}) = c^{I_1}_{o_1}(u) + c^{I_2}_{o_2}(v).$$

Details of the proof of Proposition 1.2 is in subsection 3.6.

Let $I_1 = \mathcal{O}_o$ and $I_2 = \mathcal{O}_o$. Using Proposition 1.1, we generalize Theorem 1.1 as follows

Theorem 1.2. *Let u and v be plurisubharmonic functions on Δ^n. Then*

$$c_o(\max\{u, v\}) \leq c_o(u) + c_o(v).$$

1.3. **Problems about sharp equality conditions.**

Let $n \geq 2$. Let $u = \log(\sum_{1\leq j\leq l} |z_j|^2)^{1/2}$. Note that $l > k \Rightarrow c_o(u) = l > k = c_o(u|_H)$, and $l \leq k \Rightarrow c_o(u) = l = c_{o'}(u|_H)$. Then it is natural to consider the following problem about the sharp equality condition in the restriction formula on jumping numbers:

Problem 1.1. *Let I be a coherent ideal on $\mathcal{O}_{o'}$. Suppose that*

$$c^I_{o'}(u|_H) = \sup\{c^{\tilde{I}}_o(u)|\tilde{I} \subseteq \mathcal{O}_o \ \&\ \tilde{I}|_H = I\} =: c. \tag{1.1}$$

Let $A = Supp(\mathcal{O}/\mathcal{I}(cu))$. Can one obtain that

$$dim_o A \geq n - k? \tag{1.2}$$

For the case $I = \mathcal{O}_{o'}$ and $(k, n) = (1, 2)$, Problem 1.1 was solved by Blel-Mimouni ([4]) and Favre-Jonsson ([18]).

For the case $I=\mathcal{O}_{o'}$ and $(k,n)=(1,n)$, Problem 1.1 was solved in [28].

Recently, combining with the recent result in [15] and Proposition 3.1 in [28], Rashkovskii [37] reproved the above result in [28].

It is natural to ask whether the more precise condition $\dim_o A = n-k+\dim_o(A\cap H)$ holds? However, the following example tells us that the above condition may not hold for general I.

Example 1. When $n=4$, $k=3$, $H=\{z_4=0\}$, $I=(z_1)_{o'}$, $u=2\log(|z_2|+|z_3|)+2\log(|z_1|+|z_4|)$, then $c^{I}_{o'}(u|_H)=1=\sup_{\tilde{I}}\{c^{\tilde{I}}_o(u)\}$, $A=(\{z_2=z_3=0\}\cup\{z_1=z_4=0\})$, and $\dim_o A=2<3=4-3+2=n-k+\dim_o(A\cap H)$.

It is known that for the case $k=1$ and any n, one can obtain the regularity of (A,o) (see [28]). Then it is natural to consider the regularity of (A,o) for general k. However, the following example tells us that the regularity may not hold for general I:

Example 2. When $n=2$, $k=1$, $H=\{z_2=0\}$, $I=(z_1)_{o'}$, $u=\log|z_1|+\log|z_1+z_2|$, then $c^{I}_{o'}(u|_H)=1=\sup_{\tilde{I}}\{c^{\tilde{I}}_o(u)\}$, $A=(\{z_1=0\}\cup\{z_1+z_2=0\})$, $A\cap H=\{o\}$, and (A,o) is not regular. When $n=2$, $u=\log|z_2-z_1^2|$ and $(H=\{z_2=0\})$, one can obtain that $c_{o'}(u|_H)=1/2<1=c_o(u)$.

By Example 1 and Example 2, it is natural to ask

Problem 1.2. *Assume that $c_{o'}(u|_H)=c_o(u)$.*

(1) Can one obtain $\dim_o A=n-k+\dim_o(A\cap H)$?

(2) If $(A\cap H,o)$ is regular, can one obtain that (A,o) is regular and $\dim(T_{A,o}+T_{H,o})=n$?

Note that
(a) if $u=v=\log|z|$, then $c_o(\max\{u,v\})=\frac{1}{n}=\frac{2}{n}<c_o(u)+c_o(v)$;
(b) if $n\geq 2$, $u=\log|z'|$ and $v=\log|z''|$, then $c_o(\max\{u,v\})=n=k+(n-k)=c_o(u)+c_o(v)$, where $z'=(z_1,\cdots,z_k)$ and $z''=(z_{k+1},\cdots,z_n)$.

Therefore it is natural to consider the following problem about the sharp equality condition in the generalized version of the fundamental subadditivity property of complex singularity exponents:

Problem 1.3. *Let u and v be plurisubharmonic functions on Δ^n. Let $c=c_o(u)+c_o(v)$. Let $A_1=V(\mathcal{I}(cu))$ and $A_2=V(\mathcal{I}(cv))$. If*

$$c_o(\max\{u,v\})=c, \tag{1.3}$$

can one obtain that

$$dim_oA_1+dim_oA_2\geq n? \tag{1.4}$$

2. Main results and applications

In this section, we present the main results of the present paper.

2.1. Main theorem: the solution of Problem 1.1.

In this section, we solve Problem 1.1

Theorem 2.1. *(main theorem) Suppose that equality 1.1 holds. Then inequality 1.2 holds.*

Remark 2.1. *Note that the points in $A \cap H$ are not considered in the proof of Theorem 2.1, then we obtain a more subtle conclusion:*

$$dim_o(A \setminus H) \geq n - k.$$

Let $I = \mathcal{O}_{o'}$, then we obtain the following corollary of Theorem 2.1

Corollary 2.1. *Suppose that $c_{o'}(u|_H) = c_o(u) =: c$. Then we have*

$$dim_o A \geq n - k.$$

When $k = 1$, one can obtain that

$$A = \{z | c_z(u) \leq c_o(u)\}$$

is regular at o by Siu's decomposition of positive closed currents ([40], see also [10, 9]). For details we refer to [28].

However, when $k > 1$ and $n > 2$, $\{z|c_z(u) \leq c_o(u)\}$ may not be regular at o, e.g., let $u := \log|z_1| + \log|z_2|$, $(k, n) = (2, 3)$ and $H = \{z_3 = 0\}$, then $\{z|c_z(u) \leq c_o(u)\} = (\{z_1 = 0\} \cup \{z_2 = 0\})$, and $c_o(u) = c_{o'}(u|_H) = 1$.

2.2. **Applications of the main theorem: the solutions of Problem 1.2 and Problem 1.3.**

Using Corollary 2.1, we give an affirmative answer to Problem 1.2 (1) by the following general result

Theorem 2.2. *Assume that* $\dim((A \cap H, o) \setminus (V(I), o)) = \dim_o(A \cap H)$. *If equality 1.1 holds, then* $\dim_o A = n - k + \dim_o(A \cap H)$. *Especially if* $c_{o'}(u|_H) = c_o(u)$, *then* $\dim_o A = n - k + \dim_o(A \cap H)$.

Using Theorem 2.2, we give an affirmative answer to Problem 1.2 (2).

Theorem 2.3. *Let $H = \{z_{k+1} = \cdots = z_n = 0\}$. If $c_{o'}(u|_H) = c_o(u)$, then the following statements are equivalent*

(1) *$(A \cap H, o)$ is regular;*

(2) *there exist coordinates $(w_1, \cdots, w_k, z_{k+1}, \cdots, z_n)$ near o and $l \in \{1, \cdots, k\}$ such that $(A, o) = (w_1 = \cdots = w_l = 0, o)$;*

(3) *there exist coordinates $(w_1, \cdots, w_k, z_{k+1}, \cdots, z_n)$ near o and $l \in \{1, \cdots, k\}$ such that $\mathcal{I}(c_o(u)u)_o = (w_1, \cdots, w_l)_o$.*

In the following part of this subsection, we solve Problem 1.3.

Denote by H the diagonal of $\Delta^n \times \Delta^n$. Using Proposition 1.2 ($\Omega_1 \sim \Delta^n$, $\Omega_2 \sim \Delta^n$, $I_1 \sim \mathcal{O}_o$, $I_2 \sim \mathcal{O}_o$) and Corollary 2.1 ($u \sim \max\{u \circ \pi_1, v \circ \pi_2\}$, $n \sim 2n$, $k \sim n$), where $\sim$ means that the former is replaced by the latter, we obtain that

$$dim_{(o,o)} Supp(\mathcal{O}/\mathcal{I}(c \max\{u \circ \pi_1, v \circ \pi_2\}) \geq n.$$

Note that

$$Supp(\mathcal{O}/\mathcal{I}(c \max\{u \circ \pi_1, v \circ \pi_2\}) \subseteq Supp(\mathcal{O}/\mathcal{I}(cu \circ \pi_1)) \cap Supp(\mathcal{O}/\mathcal{I}(cv \circ \pi_2)),$$

then we give an affirmative answer to Problem 1.3:

Theorem 2.4. *If equality 1.3 holds, then inequality 1.4 holds.*

Using Theorem 2.2, we present the following remark of Theorem 2.4

Remark 2.2. *If equality 1.3 holds, then* $\dim_o A_1 + \dim_o A_2 \geq n + \dim_o B$, *where* $B = \{z | c_z(u) + c_z(v) \leq c\}$ *is an analytic subset on $A_1 \cap A_2$ (see subsection 5.3).*

Let $n=2$, $u=\log|z_1|$, $v=\log|z_1-z_2^2|$. As $(|z_1|+|z_2^2|)/6\leq\max\{|z_1|,|z_1-z_2^2|\}\leq 6(|z_1|+|z_2^2|)$, it is clear that $c_o(\max\{u,v\})=1+1/2<2=c$, $A_1\cap A_2=\{o\}$. Then it is natural to consider the transversality between A_1 and A_2.

Using Theorem 2.3, we present the following sharp equality condition in Theorem 1.2 by giving the regularity of A_1 and A_2 and the transversality between A_1 and A_2.

Proposition 2.1. *Assume that (A_1,o) and (A_2,o) are both irreducible such that $(B,o)=(A_1\cap A_2,o)$, which is regular. If equality 1.3 holds, then both (A_1,o) and (A_2,o) are regular such that* $\dim(T_{A_1,o}+T_{A_2,o})=n$.

2.3. **Two sharp relations on jumping numbers and application.**

2.3.1. *A sharp upper bound of jumping numbers.*

Let $I\subseteq\mathcal{O}_o$ and $IJ\subseteq\mathcal{I}(c_o^I(u)u)_o$ ($\Leftrightarrow c_o^{IJ}(u)>c_o^I(u)$) be coherent ideals. Using the strong openness property, we obtain the following inequality on jumping numbers:

Theorem 2.5. $\frac{c_o^I(u)}{c_o^{IJ}(u)-c_o^I(u)}\geq c_o^I(\log|J|)$.

Given a coherent ideal $J\subseteq\mathcal{O}_o$. Letting $I=\mathcal{O}_o$, we obtain the following sharp upper bound of the jumping numbers represented by the complex singularity exponents:

Corollary 2.2. $c_o^J(u)\leq\frac{c_o(u)}{c_o(\log|J|)}+c_o(u)$.

The following remark illustrates the sharpness of Corollary 2.2:

Remark 2.3. *Let $(z_1,\cdots,z_n)$ be the coordinates of $\mathbb{C}^n$. Suppose $u:=c\log|z_n|$ and $J=(z_n^k)$. Then we have $c_o^J(u)=\frac{k+1}{c}$, $c_o(\log|J|)=\frac{1}{k}$ and $c_o(u)=\frac{1}{c}$. This gives the sharpness of Corollary 2.2.*

More general, when J is principal ideal (i.e., $J=(f)_o$), and $u=c\log|f|$, then we have $c_o^J(u)=\frac{1+c_o(\log|f|)}{c}$ and $c_o(u)=\frac{c_o(\log|f|)}{c}$. This implies the sharpness of Corollary 2.2.

2.3.2. *A sharp inequality for jumping numbers and their slice restrictions.*

In this subsection, we present the following sharp inequality for jumping numbers and their restrictions on hyperplanes.

Theorem 2.6. *Let φ be a plurisubharmonic function near the $o\in\mathbb{C}^n$, and $h:=z_n$. Let $H:=\{z_n=0\}$, and let $I\subset\mathcal{O}_{o'}$ be a coherent ideal, where o' is the origin in H. Let $b_0:=\sup\{c_o^{\tilde{I}}(\varphi)|\tilde{I}\subseteq\mathcal{O}_o\&\tilde{I}|_H=I\}$, and let $b_1:=\inf\{c_o^{\tilde{I}h}(\varphi)-c_o^{\tilde{I}}(\varphi)|\tilde{I}\subseteq\mathcal{O}_o\&\tilde{I}|_H=I\}$. Then*

$$b_0-c_{o'}^I(\varphi|_H)\geq b_1. \tag{2.1}$$

The sharpness of Theorem 2.6 is illustrated as follows

Remark 2.4. *Let $(k,n)=(1,2)$ ($H=\{z_2=0\}$). Let $I=(z_1)_{o'}$, and let $\varphi=\log|z|$. Then $c_{o'}^I(\varphi|_H)=2$ and $c_o^{\tilde{I}}(\varphi)=3$ for any $\tilde{I}$ (consider $|z|^2$ instead of $|\tilde{I}|^2$ and by symmetry), then $b_0=3$.*

Note that $\log|z|^4\geq\log|\tilde{I}h|^2+O(1)$. Then it follows that $c_o^{\tilde{I}h}(\varphi)\geq 4$ for any $\tilde{I}$, i.e. $b_1\geq 4-3=1$. By equality 2.1 and $b_0-c_{o'}^I(\varphi|_H)=3-2=1$, it follows that $1=b_0-c_{o'}^I(\varphi|_H)\geq b_1\geq 1$. Then we obtain the sharpness of Theorem 2.6.

Let $I=\mathcal{O}_{o'}$, then $\tilde{I}=\mathcal{O}_o$. By Theorem 2.6, it follows that

$$2c_o(\varphi)-c_o^h(\varphi)\geq c_{o'}(\varphi|_H). \tag{2.2}$$

2.3.3. *A slicing result on complex singularity exponents and an application of Theorem 2.6.*

Let $C(V_k):=c_{o'}(u|_{V_k})$ be a function on the Grassmannian $G(k,n)$ of k- dimensional linear subspaces V_k in $\mathbb{C}^n$, where $o'\in V_k$ is the origin.

Stimulated by Siu's slicing theorem on Lelong numbers [40], one can reformulate a slicing result on complex singularity exponents, which is implied by the combination of Berndtsson's log-subharmonicity of Bergman kernels [2] and solution of the openness conjecture:

Remark 2.5. *There exists $c_k\in\mathbb{R}^+\cup\{+\infty\}$ such that $C(V_k)\equiv c$ almost everywhere in the sense of the unique $U(n)$-invariant measure of mass 1 on the Grassmannian $G(k,n)$. Moveover c_k is the upper bound of $c_{o'}(u|_{V_k})$ for any V_k (details see Lemma 3.6).*

When $k=1$, it follows that $c_{o'}(V_k)=\frac{1}{\nu(u|_{V_k},o')}$, where $\nu(u|_{V_k},o')$ is the Lelong number of $u|_{V_k}$ at o'. Then Remark 2.5 degenerates to Siu's slicing theorem on Lelong numbers [40] (see also [10]) when $k=1$.

Using Theorem 2.6, we obtain the following sharp decreasing property of the intervals between consecutive c_k:

Corollary 2.3. *$c_k-c_{k-1}\geq c_{k+1}-c_k$ holds for any $k\in\{2,\cdots,n-1\}$.*

The sharpness of Corollary 2.3 can be seen as follows:

Remark 2.6. *Let $\varphi=\log|z|$, then $c_k=k$.*

3. Some preparatory results

In this section, we recall and present some preparatory results for the proof of the main theorem and applications.

3.1. Ohsawa-Takegoshi L^2 extension theorem.

We recall the original form of Ohsawa-Takegoshi L^2 extension theorem as follows:

Theorem 3.1. *([36], see also [41, 1, 7], etc.) Let D be a bounded pseudo-convex domain in $\mathbb{C}^n$. Let u be a plurisubharmonic function on D. Let H be an m-dimensional complex plane in $\mathbb{C}^n$. Then for any holomorphic function on $H\cap D$ satisying*

$$\int_{H\cap D}|f|^2e^{-2u}d\lambda_H<+\infty,$$

there exists a holomorphic function F on D satisfying $F|_{H\cap D}=f$, and

$$\int_D|F|^2e^{-2u}d\lambda_n\leq C_D\int_{H\cap D}|f|^2e^{-2u}d\lambda_H,$$

where C_D only depends on the diameter of D and m, and $d\lambda_H$ is the Lebesgue measure on H.

In [34, 7], Ohsawa-Takegoshi L^2 extension theorem has been modified as follows.

Theorem 3.2. *([34, 7], see also [10, 9]) Let D be a bounded pseudo-convex domain in $\mathbb{C}^{k+1}$. Let u be a plurisubharmonic function on D. Let $H=\{z_{k+1}=0\}$ be a complex hyperplane in $\mathbb{C}^{k+1}$. Then for any holomorphic function on $H\cap D$ satisying*

$$\int_{H\cap D}|f|^2e^{-2u}d\lambda_H<+\infty,$$

there exists a holomorphic function F on D satisfying $F|_{H\cap D}=f$, and

$$\int_D|F|^2e^{-2u-2a\log|z_{k+1}|}d\lambda_n\leq C_{D,a}\int_{H\cap D}|f|^2e^{-2u}d\lambda_H,$$

where $a\in[0,1)$, $C_{D,a}$ only depends on the diameter of D and a, and $d\lambda_H$ is the Lebesgue measure on H.

For the optimal estimate versions in general settings of Theorem 3.2 and their applications, the reader is referred to [21, 22, 23, 27, 28].

Following the symbols in Theorem 3.1, there is a local version of Theorem 3.1

Remark 3.1. *(see [36], see also [14]) For any germ of holomorphic function f on $o\in H\cap D$ satisfying $|f|^2e^{-2u|_H}$ is locally integrable near o, there exists a germ of holomorphic function F on $o\in D$ satisfying $F|_{H\cap D}=f$, and $|F|^2e^{-2u}$ is locally integrable near o.*

3.2. **Strict inequality about jumping numbers.**

Let $\Omega\ni o$ be a domain in $\mathbb{C}^n$, and let $k=n-1$ and $H=\{z_n=0\}$. Let $I\subseteq O_{o'}$ be a coherent ideal, and let v be a plurisubharmonic function on Δ^{k+1} with coordinates $(z_1,\cdots,z_k,z_{k+1})$, where $o'\in H$ is the origin.

Using Theorem 3.2, we obtain the following

Lemma 3.1. *Let I be a coherent ideal on $\mathcal{O}_{o'}$. If $c^I_{o'}(v|_H)=1$, then for any $N>0$,*

$$\sup_{\tilde{I}|_H=I}\{c^{\tilde{I}}_o(\frac{1}{2}\log(e^{2v}+|z_{k+1}|^{2N}))\}>1$$

holds, where $\tilde{I}\subseteq\mathcal{O}_o$ is a coherent ideal.

Proof. By Hölder inequality, it follows that $e^{2(1-\varepsilon)v}|z_{k+1}|^{2\varepsilon N}\leq(1-\varepsilon)e^{2v}+\varepsilon|z_{k+1}|^{2N}$, which implies

$$\frac{1}{e^{2v}+|z_{k+1}|^{2N}}\leq e^{-2(1-\varepsilon)v}|z_{k+1}|^{-2\varepsilon N},\tag{3.1}$$

where $\varepsilon\in(0,1)$.

As $c^I_{o'}(v|_H)=1$, then $|I|^2e^{-2(1-\varepsilon)v|_H}$ is integrable near o'.

By Theorem 3.2 ($u\sim(1-\varepsilon)v$, $a\sim\varepsilon N$, $f\sim I$) and choosing $\varepsilon\in(0,\frac{1}{N})$, it follows that there exists $\tilde{I}$ such that $|\tilde{I}|^2e^{-2(1-\varepsilon)v}|z_{k+1}|^{-2\varepsilon N}$ is locally integrable near o.

Using inequality 3.1, we obtain that $|\tilde{I}|^2\frac{1}{e^{2v}+|z_{k+1}|^{2N}}$ is locally integrable near o.

Using the strong openness property, we obtain the present lemma. □

Note that $c_o(\frac{1}{2}\log(e^{2v}+|z_{k+1}|^{2N}))=c_o(\max\{v,N\log|z_{k+1}|\})$ for any $N>0$ ($\Leftarrow$ $e^{2\max\{v,N\log|z_{k+1}|\}}\leq e^{2v}+|z_{k+1}|^{2N}\leq 2e^{2\max\{v,N\log|z_{k+1}|\}}$), then it follows that

Corollary 3.1. *If $c^I_{o'}(v|_H)=1$, then*

$$\sup_{\tilde{I}|_H=I}\{c^{\tilde{I}}_o(\max\{2v,\log|z_{k+1}|^N\})\}>1$$

for any $N > 0$.

After reading the earlier version of the present article, Demailly kindly pointed out that one can obtain an effectiveness result of $\sup_{\tilde{I}|_H=I}\{c_o^{\tilde{I}}(\frac{1}{2}\log(e^{2v}+|z_{k+1}|^{2N}))\}$ by using the same method as above, which can deduce the present lemma directly without using the strong openness property. The details are as follows:

Lemma 3.2. *If* $c_{o'}^I(\varphi|_H) = b > 0$,

$$\sup_{\tilde{I}|_H=I}\{c_o^{\tilde{I}}(\frac{1}{2}\log(e^{2\varphi}+|z_{k+1}|^{2\frac{N-1}{b}}))\} \geq \frac{bN}{N-1}$$

holds for any $N > 1$, *where ideal* $\tilde{I} \subseteq \mathcal{O}_o$ *is coherent.*

Proof. By Hölder inequality ($ta + (1-t)b \leq a^t + b^{1-t}$, where $t \in (0,1)$ and $a > 0$, $b > 0$), it follows that $e^{2\varepsilon b\varphi}|z_{k+1}|^{2\varepsilon} \leq \frac{N-1}{N}e^{2\varepsilon\frac{bN}{N-1}\varphi} + \frac{1}{N}|z_{k+1}|^{2\varepsilon N}$ ($t \sim \frac{N-1}{N}$, $a \sim e^{2\varepsilon\frac{bN}{N-1}\varphi}$, $b \sim |z_{k+1}|^{2\varepsilon N}$), which implies

$$\frac{1}{e^{2\varepsilon\frac{bN}{N-1}\varphi} + |z_{k+1}|^{2\varepsilon N}} \leq e^{2\varepsilon b\varphi}|z_{k+1}|^{2\varepsilon}, \tag{3.2}$$

where $\varepsilon \in (0,1)$.

As $c_{o'}^I(\varphi|_H) = b$, then $|I|^2e^{-2\varepsilon b\varphi|_H}$ is integrable near o'. By Theorem 3.2 ($\varphi \sim \varepsilon a\varphi$, $a \sim \varepsilon$, $f \sim I$) and choosing $\varepsilon \in (0, \frac{1}{N})$, it follows that there exists $\tilde{I}$ such that $|\tilde{I}|^2e^{-2\varepsilon b\varphi}|z_{k+1}|^{-2\varepsilon}$ is locally integrable near o. Using inequality 3.2, we obtain that $|\tilde{I}|^2\frac{1}{e^{2\varepsilon\frac{bN}{N-1}\varphi}+|z_{k+1}|^{2\varepsilon N}}$ is locally integrable near o.

Note that for any ε, N, b, there exist positive constants C_1, C_2 such that

$$C_1(e^{2\varphi}+|z_{k+1}|^{2\frac{N-1}{b}})^{\varepsilon\frac{bN}{N-1}} \leq e^{2\varepsilon\frac{bN}{N-1}\varphi} + |z_{k+1}|^{2\varepsilon N} \leq C_2(e^{2\varphi}+|z_{k+1}|^{2\frac{N-1}{b}})^{\varepsilon\frac{bN}{N-1}},$$

i.e.,

(1) if $\varepsilon\frac{bN}{N-1} \geq 1$, then $C_1 = (\frac{1}{2})^{\varepsilon\frac{bN}{N-1}}$ and $C_2 = 1$;

(2) if $\varepsilon\frac{bN}{N-1} < 1$, then $C_1 = 1$ and $C_2 = 2^{\varepsilon\frac{bN}{N-1}}$.

We prove the present lemma. □

Note that $c_o(\frac{1}{2}\log(e^{2\varphi}+|z_{k+1}|^{2\frac{N-1}{b}})) = c_o(\max\{\varphi, \frac{N-1}{b}\log|z_{k+1}|\})$ for any $N > 0$ ($\Leftarrow e^{2\max\{\varphi,\frac{N-1}{b}\log|z_{k+1}|\}} \leq e^{2\varphi} + |z_{k+1}|^{2\frac{N-1}{b}} \leq 2e^{2\max\{\varphi,\frac{N-1}{b}\log|z_{k+1}|\}}$), then it follows that

Corollary 3.2. $\sup_{\tilde{I}|_H=I}\{c_o^{\tilde{I}}(\max(\varphi, \frac{N-1}{b}\log|z_{k+1}|))\} \geq \frac{bN}{N-1}$ *holds with same symbols and conditions as in Lemma 3.2.*

3.3. **Hilbert's Nullstellensatz (complex situation) and jumping numbers.** It is well-known that the complex situation of Hilbert's Nullstellensatz is as follows (see (4.22) in [10])

Theorem 3.3. *(see* [10]*) For every ideal* $I \subset \mathcal{O}_o$, $\mathcal{J}_{V(I),o} = \sqrt{I}$, *where* $\sqrt{I}$ *is the radical of* I, *i.e. the set of germs* $f \in \mathcal{O}_o$ *such that some power* f^k *lies in* I.

The following lemma can be obtained by the definition of jumping numbers.

Lemma 3.3. *Let* $I \subseteq \mathcal{O}_o$ *be a coherent ideal, and* u *be a plurisubharmonic function near* o. *Then for any* $p < (0, c_o^I(u))$, $(\{z|c_z(u) \leq p\}, o) \subseteq (V(I), o)$ *holds.*

Lemma 3.3 implies the following

Remark 3.2. *Let $I \subseteq \mathcal{O}_o$ be a coherent ideal, and u be a plurisubharmonic function near o. Let (A, o) be a germ of analytic set such that $c_z(u) \leq c_o(u)$ for any $z \in (A, o)$ and $\dim((A, o) \setminus (V(I), o)) = \dim_o A$. Let U be a neighborhood of o small enough such that $\dim(A \cap U) = \dim_o A$. Then for any $p \in (0, c_o(u))$, $\dim((A \cap U) \setminus \{z|c_z(u) \leq p\}) = \dim_o A$ holds. Moreover there exists $z_0 \in ((U \cap A) \setminus V(I))$ such that $\dim_{z_0} A = \dim_o A$ and $z_0 \notin (\cup_{p\in(0,c_o(u))}\{z|c_z(u) \leq p\})$, which implies $c_{z_0}(u) = c_o(u)$.*

3.4. **A useful proposition in [28] and some generalizations.**

In [28], using Demailly's idea of equisingular approximations of quasiplurisubharmonic functions (see [9], see also [11]) and the strong openness property of the multiplier ideal sheaf (see [26]), we have obtained the following proposition:

Proposition 3.1. [28] *Let D be a bounded domain in $\mathbb{C}^n$, and the origin $o \in D$. Let u be a plurisubharmonic function on D. Let (g_j) be a (finite) local basis of $\mathcal{I}(u)_o$. Then there exists $l > 1$ such that $e^{-2u} - e^{-2\max\{u, \frac{1}{l-1}\log\sum_j |g_j|\}}$ is integrable on a small enough neighborhood V_o of o.*

Given a coherent ideal $I \subseteq \mathcal{I}(u)_o$ and let (h_j) be the basis of I. Using (h_j) instead of (g_j) in the proof of Proposition 3.1 in [28], one can obtain

Remark 3.3. *For any $I \subset \mathcal{I}(u)_o$, we have $e^{-2u} - e^{-2\max\{u, \frac{1}{l-1}\log|I|\}} < +\infty$. where $|I| = \sum_j |h_j|$, and $l \in (1, c_o^I(u))$ is the positive number as in Proposition 3.1.*

Let $n = k+1$. It is well-known that if $\{z|\mathcal{I}(u)_z \neq \mathcal{O}_z\} \subset \{z_{k+1} = 0\}$, then there exists $N_0 > 0$ large enough such that $(z_{k+1}^{N_0})_o \subseteq \mathcal{I}(u)_o$.

Corollary 3.3. *If $c_o^J(u) \leq 1$ ($\Rightarrow$ "$|J|^2e^{-2u}$ is not integrable near o" by using the strong openness property), then*

$$c_o^J(\max\{u, N\log|z_{k+1}|\}) \leq 1$$

for any $N \geq \frac{1}{l-1}N_0$ (independent of J), where $J \subseteq \mathcal{O}_o$ is a coherent ideal, $l \in (1, c_o^I(u))$ and $I := (z_{k+1}^{N_0})_o$. Especially, if $c_o^J(u) = 1$, then

$$c_o^J(\max\{u, N\log|z_{k+1}|\}) = 1.$$

Proof. Using Remark 3.3, one can obtain that $e^{-2u} - e^{-2\max\{u, \frac{1}{l-1}N_0\log|z_{k+1}|\}}$ is locally integrable near o by letting $I = (z_{k+1}^{N_0})_o$.

As $|J|^2e^{-2u}$ is not locally integrable near o ($\Leftarrow c_o^J(u) = 1$), then it follows that $e^{-2\max\{u, \frac{1}{l-1}N_0\log|z_{k+1}|\}}$ is not locally integrable near o, which implies

$$c_o^J(\max\{u, \frac{1}{l-1}N_0\log|z_{k+1}|\}) \leq 1.$$

Since $c_o^J(\max\{u, \frac{1}{l-1}N_0\log|z_{k+1}|\}) \geq c_o^J(u) = 1$ ($\Leftarrow \max\{u, \frac{1}{l-1}N_0\log|z_{k+1}|\} \geq u$), then it follows that $c_o^J(\max\{u, \frac{1}{l-1}N_0\log|z_{k+1}|\}) = 1$.

As $N\log|z_{k+1}| \leq \frac{1}{l-1}N_0\log|z_{k+1}|$, then it follows that $u \leq \max\{u, N\log|z_{k+1}|\} \leq \max\{u, \frac{1}{l-1}N_0\log|z_{k+1}|\}$, which implies

$$c_o^J(u) \leq c_o(\max\{u, N\log|z_{k+1}|\}) \leq c_o^J(\max\{u, \frac{1}{l-1}N_0\log|z_{k+1}|\}).$$

Note that $c_o^J(u) = c_o^J(\max\{u, \frac{1}{l-1}N_0\log|z_{k+1}|\}) = 1$, then we obtain the present corollary. $\square$

Let $I \subseteq \mathcal{O}_o$. Let J be an coherent ideal satisfying $IJ \subseteq \mathcal{I}(c_o^I(\varphi)\varphi)_o$ ($\Leftrightarrow IJ \subseteq \mathcal{I}_o(c_o^I(\varphi)\varphi)$). Let $\{f_j\}_{j=1,2,\cdots,s}$ be a local basis of J_o. Denoted by $|J| := \sum_{i=1}^s |f_i|$. Let $\tilde{\varphi}_l := \max\{c_o^I(\varphi)\varphi, \frac{1}{l-1}\log|J|\}$.

If $\psi_1 - C \leq \psi \leq \psi_1 + C$, then

$$\begin{aligned} \max\{\varphi,\psi_1\} - C &\leq \max\{\varphi,\psi_1 - C\} \\ &\leq \max\{\varphi,\psi\} \\ &\leq \max\{\varphi,\psi_1 + C\} \leq \max\{\varphi,\psi_1\} + C, \end{aligned} \tag{3.3}$$

where φ, ψ_1 and ψ are plurisubharmonic functions.

By inequality 3.3 ($\varphi \sim c_o^I(\varphi)\varphi$, $\psi \sim \frac{1}{l-1}\log|J|$), it follows that

$$c_o^I(\max\{c_o^I(\varphi)\varphi, \frac{1}{l-1}\log|J|\})$$

is well-defined for any basis of J. The proof of Proposition 2.1 in [28] also implies the following

Remark 3.4. *For any $l \in (1, \frac{c_o^{IJ}(\varphi)}{c_o^I(\varphi)}]$, we have $c_o^I(\tilde{\varphi}_l) = 1$.*

Proof. For the convenience of the readers, we recall the proof in [28] with subtle modifications as follows:

It is clear that there exists a small enough neighborhood $V_1 \ni o$ such that

$$\int_{V_1} |IJ|^2 e^{-2c_o^I(\varphi)\varphi} < \infty. \tag{3.4}$$

Given any real number $l \in (1, \frac{c_o^{IJ}(\varphi)}{c_o^I(\varphi)})$, by the strong openness property, there exists a small neighborhood V_2 of o such that

$$\int_{V_2} |IJ|^2 e^{-2lc_o^I(\varphi)\varphi} < \infty. \tag{3.5}$$

Then

$$\begin{aligned} \int_{V_2} |I|^2 (e^{-2c_o^I(\varphi)\varphi} - e^{-2\tilde{\varphi}_l}) &\leq \int_{\{\varphi < \frac{1}{(l-1)c_o^I(\varphi)}\log|J|\}\cap V_2} |I|^2 e^{-2c_o^I(\varphi)\varphi} \\ &= \int_{\{\varphi < \frac{1}{(l-1)c_o^I(\varphi)}\log|J|\}\cap V_2} |I|^2 e^{2(l-1)c_o^I(\varphi)\varphi - 2lc_o^I(\varphi)\varphi} \\ &\leq \int_{\{\varphi < \frac{1}{(l-1)c_o^I(\varphi)}\log|J|\}\cap V_2} |I|^2 e^{2\log|J| - 2lc_o^I(\varphi)\varphi} \\ &\leq \int_{V_2} |I|^2|J|^2 e^{-2lc_o^I(\varphi)\varphi} < +\infty, \end{aligned} \tag{3.6}$$

where the last inequality follows from inequality 3.5.

As $|I|^2(e^{-2c_o^I(\varphi)\varphi} - e^{-2\tilde{\varphi}_l})$ is integrable near o, and $|I|^2 e^{-2c_o^I(\varphi)\varphi}$ is not integrable near o, it follows that $|I|^2 e^{-2\tilde{\varphi}_l}$ is not integrable near o, which implies $c_o^I(\tilde{\varphi}_l) \leq 1$.

As $c_o^I(\varphi)\varphi \leq \tilde{\varphi}_l$, it follows that

$$|I|^2 e^{-2cc_o^I(\varphi)\varphi} \geq |I|^2 e^{-2c\tilde{\varphi}_l}$$

for any $c > 0$. When $c \in (0,1)$, by the definition of jumping numbers, it follows that $|I|^2 e^{-2cc^I_o(\varphi)\varphi}$ is locally integrable near o, which implies $|I|^2 e^{-2c\tilde{\varphi}_l}$ is locally integrable near o, i.e., $c^I_o(\tilde{\varphi}_l) \geq 1$. Then we have $c^I_o(\tilde{\varphi}_l) = 1$. $\square$

We recall a consequence of Proposition 3.1 as follows

Remark 3.5. *([28]) $\tilde{u}$ (as in Proposition 3.1) has the following properties*

(1) *for any $z \in (\{z|c_z(u) \leq 1\}, o) = (V(\mathcal{I}(u)), o)$, inequality $c_z(u) \leq c_z(\tilde{u}) \leq 1$ holds;*

(2) *if $c_{z_0}(u) = 1$, then $c_{z_0}(\tilde{u}) = 1$, where $z_0 \in (\{z|c_z(u) \leq 1\}, o)$.*

Let $I \subseteq \mathcal{O}_o$ be a coherent ideal, and let u be a plurisubharmonic function near o. We present the following consequence of Remark 3.5 about the integrability of the ideals related to weight of jumping number one.

Proposition 3.2. *Let $J \subseteq \mathcal{O}_o$ be a coherent ideal. Assume that $c^I_o(u) = 1$. If $(V(\mathcal{I}(u)), o) \subseteq (V(J), o)$, then $|I|^2|J|^{2\varepsilon}e^{-2u}$ is locally integrable near o for any $\varepsilon > 0$.*

After the present article has been written, Demailly kindly shared his manuscript [12] with the first author of the present article, which includes Proposition 3.2 (Lemma (4.2) in [12]) as a consequence of the strong openness property of the multiplier ideal (see [26]).

Proof. (proof of Proposition 3.2) Let $J_0 \subseteq \mathcal{O}_o$ be a coherent ideal satisfying $(V(J_0), o) \supseteq (V(\mathcal{I}(u)), o)$. By Theorem 3.3 ($I \sim \mathcal{I}(u)_o$), it follows that there exists large enough positive integer N such that $J_0^N \subseteq \mathcal{I}(u)_o$.

By Remark 3.5, it follows that exist $p_0 > 0$ large enough such that $e^{-2u} - e^{-2\max\{u, p_0 \log|J_0|\}}$ is locally integrable near o.

It suffices to prove that $|I|^2|J_0|^{2\varepsilon}e^{-2\max\{u,p_0\log|J_0|\}}$ is locally integrable near o for small enough $\varepsilon > 0$. We prove the above statement by contradiction: If not, then there exists $\varepsilon_0 > 0$, such that $|I|^2|J_0|^{2\varepsilon_0}e^{-2\max\{u,p_0\log|J_0|\}}$ is not locally integrable near o. Note that $\varepsilon_0 \log|J_0| \leq \frac{\varepsilon_0}{p_0}\max\{u, p_0\log|J_0|\}$, then it follows that $|I|^2 e^{-2(1-\frac{\varepsilon_0}{p_0})\max\{u,p_0\log|J_0|\}}$ is not locally integrable near o. Note that $u \leq \max\{u, p_0 \log|J_0|\}$, then it follows that $|I|^2 e^{-2(1-\frac{\varepsilon_0}{p_0})u}$ is not locally integrable near o, which contradicts $c^I_o(u) = 1$. Then we prove Proposition 3.2. $\square$

Let $I = \mathcal{O}_o$, $\varepsilon = 1$. Using Proposition 3.2, we obtain the following result.

Corollary 3.4. *Let $J \subseteq \mathcal{O}_o$ be a coherent ideal. Assume that $c_o(u) = 1$. Then the following two statements are equivalent*

(1) *$(V(J), o) \supseteq (V(\mathcal{I}(u)), o)$;*

(2) *$|J|^2 e^{-2u}$ is locally integrable near o, i.e. $J \subseteq \mathcal{I}(u)_o$.*

3.5. **Measures along the fibres.**

Let $X := \{z_{k+1} = \cdots = z_n = 0\}$. Consider a map q from $\mathbb{C}^n \setminus X$ to $\mathbb{CP}^{n-k-1}$: $q(z_1, \cdots, z_n) = (z_{k+1} : \cdots : z_n)$.

Let Y be an analytic set in $\mathbb{B}^n$ whose complex dimension is smaller than $n - k$. By the same proof as that of Lemma 2.8 in [28] (the methods can be found in [20] and [6]), one can obtain that

Lemma 3.4. *For almost all $(z_{k+1}:\cdots:z_n)$, the complex dimension of $q^{-1}(z_{k+1}:\cdots:z_n)\cap Y$ is zero, i.e., $(\overline{q^{-1}(z_{k+1}:\cdots:z_n)}\cap Y,o)=(X\cap Y,o)$.*

Proof. Note that the $2(n-k)-2$ dimensional Hausdorff measure on $\mathbb{CP}^{n-k-1}$ is positive, and $2(n-k)$ dimensional Hausdorff measure of Y is zero, then it follows that for almost all $(z_{k+1}:\cdots:z_n)$, the 2 dimensional Hausdorff measure of $q^{-1}(z_{k+1}:\cdots:z_n)\cap Y$ is zero, i.e., the complex dimension of $q^{-1}(z_{k+1}:\cdots:z_n)\cap Y$ is zero. Then we obtain the present lemma. □

3.6. **Proof of Proposition 1.2.**

By the definition of jumping numbers, it follows that for any $\varepsilon>0$, there exists a neighborhood U_ε of o and $C_\varepsilon>0$ such that

$$r^{-2(c_o^I(u)-\varepsilon)}\int_{\Delta^n}\mathbb{I}_{\{u<\log r\}\cap U_\varepsilon}|I|^2d\lambda_n<C_\varepsilon$$

holds for any $r>0$, which implies

$$\liminf_{r\to 0^+}\frac{\log(\int_{\Delta^n}\mathbb{I}_{\{u<\log r\}\cap U_\varepsilon}|I|^2d\lambda_n)}{2\log r}\geq c_o^I(u)-\varepsilon. \tag{3.7}$$

We recall our solution of a conjecture posed by Demailly-Kollar [14] (which means that $\liminf_{r\to 0^+}(-r^{2c_o^I(u)}\int_{\Delta^n}\mathbb{I}_{\{u<\log r\}\cap U}d\lambda_n)>0$, holds) as follows

Theorem 3.4. [25, 27] *Let u be a plurisubharmonic function on $\Delta^n\subset\mathbb{C}^n$ and I be a coherent ideal in $\mathcal{O}_o$. Then for any neighborhood U of o, there exists $C_\varepsilon>0$ such that*

$$(-r^{2c_o^I(u)}\int_{\Delta^n}\mathbb{I}_{\{u<\log r\}\cap U}|I|^2d\lambda_n)>C_\varepsilon.$$

By Theorem 3.4, it follows that for any neighborhood U of o,

$$\limsup_{r\to 0^+}\frac{\log(\int_{\Delta^n}\mathbb{I}_{\{u<\log r\}\cap U}|I|^2d\lambda_n)}{2\log r}\leq c_o^I(u) \tag{3.8}$$

holds.

As $\{\max\{u\circ\pi_1,v\circ\pi_2\}<\log r\}\cap(\pi_1^{-1}(U)\cap\pi_2^{-1}(V))=(\{u<\log r\}\cap U)\times(\{v<\log r\}\cap V)$, then it follows that

$$\begin{aligned}&\int_{\Delta^n\times\Delta^n}\mathbb{I}_{\{\max\{u\circ\pi_1,v\circ\pi_2\}<\log r\}\cap(\pi_1^{-1}(U)\cap\pi_2^{-1}(V))}(|\pi_1^*I|\times|\pi_2^*J|)^2d\lambda_{2n})\\&=\int_{\Delta^n}\mathbb{I}_{\{u<\log r\}\cap U}|I|^2d\lambda_n\times\int_{\Delta^n}\mathbb{I}_{\{v<\log r\}\cap V}|J|^2d\lambda_n,\end{aligned} \tag{3.9}$$

where U and V are neighborhoods of $o\in\mathbb{C}^n$, I and J are coherent ideals in $\mathcal{O}_o$.

By inequality 3.7, it follows that for any $\varepsilon>0$, there exist neighborhoods U_ε and V_ε of o such that

$$\liminf_{r\to 0^+}\frac{\log\int_{\Delta^n}\mathbb{I}_{\{u<\log r\}\cap U_\varepsilon}|I|^2d\lambda_n}{2\log r}\geq c_o^I(u)-\varepsilon$$

and

$$\liminf_{r\to 0^+}\frac{\log\int_{\Delta^n}\mathbb{I}_{\{v<\log r\}\cap V_\varepsilon}|J|^2d\lambda_n}{2\log r}\geq c_o^J(v)-\varepsilon.$$

By inequality 3.8, it follows that

$$\begin{aligned}
&c_o^{I\times J}(\max\{u\circ\pi_1, v\circ\pi_2\})\\
=&\limsup_{r\to 0^+}\frac{\log\int_{\Delta^n\times\Delta^n}\mathbb{I}_{\{\max\{u\circ\pi_1,v\circ\pi_2\}<\log r\}\cap(\pi_1^{-1}(U_\varepsilon)\cap\pi_2^{-1}(V_\varepsilon))}(|\pi_1^*I|\times|\pi_2^*J|)^2d\lambda_{2n}}{2\log r}\\
\geq&\liminf_{r\to 0^+}\frac{\log\int_{\Delta^n\times\Delta^n}\mathbb{I}_{\{\max\{u\circ\pi_1,v\circ\pi_2\}<\log r\}\cap(\pi_1^{-1}(U_\varepsilon)\cap\pi_2^{-1}(V_\varepsilon))}(|\pi_1^*I|\times|\pi_2^*J|)^2d\lambda_{2n}}{2\log r}\\
=&\liminf_{r\to 0^+}\frac{\log\int_{\Delta^n}\mathbb{I}_{\{u<\log r\}\cap U_\varepsilon}|I|^2d\lambda_n}{2\log r}+\liminf_{r\to 0^+}\frac{\log\int_{\Delta^n}\mathbb{I}_{\{v<\log r\}\cap V_\varepsilon}|J|^2d\lambda_n}{2\log r}\\
\geq&(c_o^I(u)-\varepsilon)+(c_o^J(v)-\varepsilon).
\end{aligned}\tag{3.10}$$

By inequality 3.7, it follows that for any $\varepsilon>0$, there exist neighborhoods U'_ε and V'_ε of o such that

$$\begin{aligned}
&\liminf_{r\to 0^+}\frac{\log\int_{\Delta^n\times\Delta^n}\mathbb{I}_{\{\max\{u\circ\pi_1,v\circ\pi_2\}<\log r\}\cap(\pi_1^{-1}(U'_\varepsilon)\cap\pi_2^{-1}(V'_\varepsilon))}(|\pi_1^*I|\times|\pi_2^*J|)^2d\lambda_{2n}}{2\log r}\\
&\qquad\geq c_o^{I\times J}(\max\{u\circ\pi_1, v\circ\pi_2\})-\varepsilon.
\end{aligned}\tag{3.11}$$

By inequality 3.8, it follows that

$$\begin{aligned}
&c_o^I(u)+c_o^J(v)\\
\geq&\limsup_{r\to 0^+}\frac{\log\int_{\Delta^n}\mathbb{I}_{\{u<\log r\}\cap U'_\varepsilon}|I|^2d\lambda_n}{2\log r}+\limsup_{r\to 0^+}\frac{\log\int_{\Delta^n}\mathbb{I}_{\{v<\log r\}\cap V'_\varepsilon}|J|^2d\lambda_n}{2\log r}\\
=&\limsup_{r\to 0^+}\frac{\log\int_{\Delta^n\times\Delta^n}\mathbb{I}_{\{\max\{u\circ\pi_1,v\circ\pi_2\}<\log r\}\cap(\pi_1^{-1}(U'_\varepsilon)\cap\pi_2^{-1}(V'_\varepsilon))}(|\pi_1^*I|\times|\pi_2^*J|)^2d\lambda_{2n}}{2\log r}\\
\geq&\liminf_{r\to 0^+}\frac{\log\int_{\Delta^n\times\Delta^n}\mathbb{I}_{\{\max\{u\circ\pi_1,v\circ\pi_2\}<\log r\}\cap(\pi_1^{-1}(U'_\varepsilon)\cap\pi_2^{-1}(V'_\varepsilon))}(|\pi_1^*I|\times|\pi_2^*J|)^2d\lambda_{2n}}{2\log r}\\
\geq&c_o^{I\times J}(\max\{u\circ\pi_1, v\circ\pi_2\})-\varepsilon.
\end{aligned}\tag{3.12}$$

Letting $\varepsilon\to 0$, using inequality 3.10 and inequality 3.12, we obtain Proposition 1.2.

3.7. **Slicing result on complex singularity exponent and subadditivity theorem on jumping numbers.**

Let v be a plurisubharmonic function on Δ^n. Let $\mathcal{H}_{2v}(\Delta^n)$ be the Hilbert space of the holomorphic function f on Δ^n satisfying (the norm) $(\int_{\Delta^n}|f|^2e^{-2v})^{1/2}<+\infty$. Let $K_{\Delta^n,2v}$ be the Bergman kernel associated with $\mathcal{H}_{2v}(\Delta^n)$.

It is easy to see that $\int_{\Delta^n_r}e^{-2v}=+\infty$ (for any $r>0$) if and only if $K_{\Delta^n,2v}(o)=0$, where o is the origin in $\mathbb{C}^n$.

By definition of $c_o(v)$, it follows that $\int_{\Delta^n_r}e^{-2v}=+\infty$ (for any $r>0$) implies $c_o(v)\leq 1$; by Berndtsson's solution of the openness conjecture, it follows that $c_o(v)\leq 1$ implies $\int_{\Delta^n_r}e^{-2v}=+\infty$ (for any $r>0$). Then one can obtain

Lemma 3.5. *$c_o(v)\leq 1$ if and only if $K_{\Delta^n,2v}(o)=0$.*

Let $p:\Delta^n\times\Delta^m\to\Delta^m$ be the projection satisfying $p(z_1,\cdots,z_n,w_1,\cdots,w_m)=(w_1,\cdots,w_m)$, where $(z_1,\cdots,z_n)$ and $(w_1,\cdots,w_m)$ are coordinates on $\mathbb{C}^n$ and $\mathbb{C}^m$.

Let u be a plurisubharmonic function on $\Delta^n\times\Delta^m$. Let K_{2cu} be the fiberwise Bergman kernel on $\Delta^n\times\Delta^m$ such that $K_{2cu}|_{p^{-1}(w)}$ is the Bergman kernel associated with the Hilbert space $\mathcal{H}_{2cu|_{p^{-1}(w)}}(p^{-1}(w))$ (see [2]).

Berndtsson's important result on log-subharmonicity of the Bergman kernels [2] tells us that $\log K_{2cu}$ is plurisubharmonic. Combining with Lemma 3.5, one can obtain

Lemma 3.6. *For any $a>0$, $\{w|c_{(o,w)'}(u|_{p^{-1}(w)})\leq a\}$ is a complete pluripolar set on Δ^m, which implies that $c_{(o,w)'}(u|_{p^{-1}(w)})$ are the same (denoted by C) for almost every w in the sense of Lebesgue measure on $\mathbb{C}^m$, $(o,w)'\in p^{-1}(w)$. Moveover $C=\sup_{w\in\Delta^m}\{c_{(o,w)'}(u|_{p^{-1}(w)})\}$.*

Proof. By Lemma 3.5 ($v=au|_{p^{-1}(w)}$), it follows that

$$\{w|c_{(o,w)'}(u|_{p^{-1}(w)})\leq a\}=\{w|\log K_{2au}(o,w)=-\infty\},$$

which implies $\{w|c_{(o,w)'}(u|_{p^{-1}(w)})\leq a\}$ is a complete pluripolar set on Δ^m.

Note that the Lebesgue measure of pluripolar set is 0 or π^m. It follows that $c_{(o,w)'}(u|_{p^{-1}(w)})$ are the same (denoted by C) for almost every w.

We prove "moreover" part by contradiction: if not, then it follows that there exists w satisfying $c_{(o,w)'}(u|_{p^{-1}(w)})>C$, which implies

$$\log K_{2Cu}(o,w)>-\infty.$$

As $\log K_{2Cu}(o,w)$ is plurisubharmonic, then it follows that there exists a neighborhood U of w such that

$$\log K_{2Cu}(o,\cdot)>-\infty$$

for almost all point in U.

Using Berndtsson's solution of the openness conjecture, one can obtain

$$c_{(o,\cdot)'}(u|_{p^{-1}(\cdot)}(o,\cdot))>C$$

holds for almost all point in U, which contradicts "$c_{(o,w)'}(u|_{p^{-1}(w)})=C$ for almost all $w\in\Delta^m$". $\square$

By the strong openness property, one can also obtain an analogue of the restriction formula for multiplier ideal,

$$c_o^I(u\circ f)\leq\sup\{c_{f(o)}^{\tilde{I}}(u)|f^*\tilde{I}=I\ \&\ \tilde{I}\subseteq\mathcal{O}_{f(o)}\},$$

which is equivalent to the comparison property on the multiplier ideals: $\mathcal{I}(u\circ f)\subseteq f^*\mathcal{I}(u)$ (see [13], see also (14.3) in [9]), where f is a holomorphic map.

In [13] (see also Theorem (14.2) in [9]), the following subadditivity theorem on jumping numbers has been presented

Theorem 3.5. *Subadditivity Theorem*
(a) $\pi_i:=\Omega_1\times\Omega_2\to\Omega_i$ $i=1,2$ the projections, and let u_i be a plurisubharmonic function on Ω_i. Then

$$\mathcal{I}(u_1\circ\pi_1+u_2\circ\pi_2)=\pi_1^*(\mathcal{I}(u_1))\cdot\pi_2^*(\mathcal{I}(u_2)).$$

(b) Let Ω be a domain and let u and v be two plurisubharmonic functions on Ω. Then

$$\mathcal{I}(u+v) \subseteq \mathcal{I}(u) \cdot \mathcal{I}(v).$$

By the strong openness property, it follows that Theorem 3.5 is equivalent to the following result:

Theorem 3.6.
(a) Let $\Omega_1 \ni o_1$ and $\Omega_2 \ni o_2$ be two domains, $\pi_i := \Omega_1 \times \Omega_2 \to \Omega_i$ $i = 1, 2$ the projections, and let u_i be a plurisubharmonic function on Ω_i. Then

$$c_{o_1 \times o_2}^{\tilde{I}}(u_1 \circ \pi_1 + u_2 \circ \pi_2) = \sup\{\min\{c_{o_1}^{J_1}(u_1), c_{o_2}^{J_2}(u_2)\} | J_1 \cdot J_2 \supseteq \tilde{I}\}, \tag{3.13}$$

where $\tilde{I}$ is a coherent ideal in $\mathcal{O}_{o_1 \times o_2}$, J_1 and J_2 are coherent ideals in $\mathcal{O}_{o_1}$ and $\mathcal{O}_{o_2}$ respectively.
(b) Let Ω be a domain, let u and v be two plurisubharmonic functions on $\Omega \ni o$. Then

$$c_o^I(u+v) \le \sup\{\min\{c_o^{I_1}(u), c_o^{I_2}(v)\} | I_1 \cdot I_2 \supseteq I\} \tag{3.14}$$

where I_1 and I_2 are coherent ideals in $\mathcal{O}_o$.

Let $\Omega_i \subset \mathbb{C}^n$ and containing the origin $o \in \mathbb{C}^n$ for any $i \in \{1, 2\}$. Let Δ be the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$. It is well-known that

Remark 3.6. *Let A_1 and A_2 be two varieties on Ω_1 and Ω_2 respectively through o. Assume that A_1 and A_2 are both regular at o. Then $\dim(T_{A_1,o} \cap T_{A_2,o}) = \dim(T_{A_1 \times A_2,(o,o)} \cap T_{\Delta,(o,o)})$.*

3.8. **Applications of the slicing result on complex singularity exponent.** Let $(z_1, \cdots, z_k)$ be the coordinates of $\mathbb{B}^{k-l} \times \mathbb{B}^l \subseteq \mathbb{C}^k$, and let $p : \mathbb{B}^{k-l} \times \mathbb{B}^l \to \mathbb{B}^{k-l}$. Let $H_1 := \{z_{k-l+1} = \cdots = z_k = 0\}$.

We present a corollary of Lemma 3.6 as follows

Corollary 3.5. *Let u be a plurisubharmonic function on $\mathbb{B}^{k-l} \times \mathbb{B}^l$. Assume that $c_z(u) \le 1$ for any $z \in H_1$ and $c_o(u) = 1$, where o is the origin in $\mathbb{B}^{k-l} \times \mathbb{B}^l$. Then for almost every $a = (a_1, \cdots, a_{k-l}) \in \mathbb{B}^{k-l}$ with respect to the Lebesgue measure on $\mathbb{C}^{k-l}$, $c_{z'_a}(u|_{L_a}) = 1$ holds, where $L_a = \{z_1 = a_1, \cdots, z_{k-l} = a_{k-l}\}$, and $z'_a \in L_a \cap H_1$ emphasizes that $c_{z'_a}(u|_{L_a})$ is computed on the submanifold L_a.*

Proof. By Lemma 3.6 and $c_o(u) = 1$, it follows that $c \ge 1$ (consider the integrability of e^{-2pu} near o, where $p < 1$ near 1, and by contradiction).

By $c_z(u) \le 1$ for any $z \in H_1$ and Proposition 1.1, it follows that $c_{z'_a}(u|_{L_a}) \le c_{z_a}(u) \le 1$ for any $z'_a \in L_a \cap H_1$. Combining $c \ge 1$, we obtain Corollary 3.5. $\square$

The following remark is the singular version of Corollary 3.5:

Remark 3.7. *Let A_3 be a reduced irreducible analytic subvariety on $\mathbb{B}^{k-l} \times \mathbb{B}^l$ through o satisfying $\dim_o A_3 = k - l$ such that*

(1) for any $a = (a_1, \cdots, a_{k-l}) \in \mathbb{B}^{k-l}$, $A_3 \cap L_a \neq \emptyset$, where $L_a = \{z_1 = a_1, \cdots, z_{k-l} = a_{k-l}\}$;

(2) there exists analytic subset $A_4 \subseteq \mathbb{B}^{k-l}$ such that any $z \in (A_3 \cap p^{-1}(\mathbb{B}^{k-l} \setminus A_4))$ is the regular point in A_3 and the noncritical point of $p|_{A_{3,reg}}$.

Let u be a plurisubharmonic function on $\mathbb{B}^{k-l} \times \mathbb{B}^l$. Assume that $c_z(u) \le 1$ for any $z \in A_3$ and $c_o(u) = 1$. Then for almost every $a = (a_1, \cdots, a_{k-l}) \in \mathbb{B}^{k-l}$ with

respect to the Lebesgue measure on $\mathbb{C}^{k-l}$, there exists $z_a \in A_3 \cap L_a$ such that equality $c_{z'_a}(u|_{L_a}) = 1$.

Proof. By Lemma 3.6, it follows that there exists $c \in (0, \infty]$. such that $c_{z'}(u|_{L_{p(z)}}) = c$ for almost every $z \in (A_3 \cap p^{-1}(\mathbb{B}^{k-l} \setminus A_4))$ with respect to the Lebesgue measure in $(A_3 \cap p^{-1}(\mathbb{B}^{k-l} \setminus A_4))$. By $c_o(u) = 1$, it follows that $c \geq 1$ (consider the integrability of e^{-2pu} near o, where $p < 1$ near 1, and by contradiction).

By Proposition 1.1, it follows that $1 \leq c \leq c_z(u|_{L_{p(z)}}) \leq c_z(u) \leq 1$ holds, for almost every $z \in (A_3 \cap p^{-1}(\mathbb{B}^{k-l} \setminus A_4))$. Then one can find $z_3 \in (A_3 \cap p^{-1}(\mathbb{B}^{k-l} \setminus A_4))$ such that $c_{z_3}(u) = c_{z'_3}(u|_{L_{p(z_3)}}) = 1$.

By Corollary 3.5 ($o \sim z_3$), it follows that for almost every $a = (a_1, \cdots, a_{k-l}) \in (\mathbb{B}^{k-l} \setminus A_4)$ with respect to the Lebesgue measure on $\mathbb{C}^{k-l}$, there exists $z_a \in A_3 \cap L_a$ such that equality $c_{z'_a}(u|_{L_a}) = 1$. As the Lebesgues measure of A_4 on $\mathbb{C}^{k-l}$ is zero, then we obtain Remark 3.7. $\square$

4. Proof of Theorem 2.1 (main theorem)

It suffices to consider the case $c^I_{o'}(u|_H) = 1$ (consider $c^I_{o'}(u|_H)u$ instead of u).

By $\sup\{c^{\tilde{I}}_o(u)|\tilde{I} \subseteq \mathcal{O}_o \ \&\ \tilde{I}|_H = I\} = 1$ ($\Rightarrow |\tilde{I}|^2 e^{-2u}$ is not locally integrable near o) and Proposition 3.1, it follows that

$$\sup\{c^{\tilde{I}}_o(\tilde{u})|\tilde{I} \subseteq \mathcal{O}_o \ \&\ \tilde{I}|_H = I\} \leq 1,$$

where $\tilde{u} := \max\{u, \frac{1}{l-1} \log \sum_j |g_j|\}$. Since $\tilde{u} \geq u$, which implies

$$\sup\{c^{\tilde{I}}_o(\tilde{u})|\tilde{I} \subseteq \mathcal{O}_o \ \&\ \tilde{I}|_H = I\} \geq \sup\{c^{\tilde{I}}_o(u)|\tilde{I} \subseteq \mathcal{O}_o \ \&\ \tilde{I}|_H = I\} = 1,$$

then it follows that $\sup\{c^{\tilde{I}}_o(\tilde{u})|\tilde{I} \subseteq \mathcal{O}_o \ \&\ \tilde{I}|_H = I\} = 1$.

By the restriction formula on jumping numbers, it follows that $\sup\{c^{\tilde{I}}_o(\tilde{u})|\tilde{I} \subseteq \mathcal{O}_o \ \&\ \tilde{I}|_H = I\} \geq c^I_{o'}(\tilde{u}|_H) \geq c^I_{o'}(u|_H) = 1$ ($\Leftarrow \tilde{u}|_H \geq u|_H$), which implies $c_{o'}(\tilde{u}|_H) = \sup\{c^{\tilde{I}}_o(\tilde{u})|\tilde{I} \subseteq \mathcal{O}_o \ \&\ \tilde{I}|_H = I\} = 1$.

Let $Y := Supp\{z|c_z(\tilde{u}) \leq 1\} = Supp(\mathcal{O}/\mathcal{I}(c^I_{o'}(u|_H)u))$. We prove Theorem 2.1 by contradiction. If not, then $dimY < n - k$. By Lemma 3.4, there exists a $k+1$ dimensional plane $H_1 \supset H$ such that $H_1 \cap Y = H \cap Y$ (without loss of generality, one can retract the Δ^n). By changing of the coordinates, we set $H_1 := \{z_{k+2} = \cdots = z_n\}$.

By the restriction formula on jumping numbers, it follows that

$$\sup\{c^{\tilde{I}}_o(\tilde{u})|\tilde{I} \subseteq \mathcal{O}_o \ \&\ \tilde{I}|_H = I\} \geq \sup\{c^{\tilde{I}}_{o''}(\tilde{u}|_{H_1})|\tilde{I} \subseteq \mathcal{O}_{o''} \ \&\ \tilde{I}|_H = I\} \geq c^I_{o'}(\tilde{u}|_H),$$

where $o'' \in H_1$ is the origin, which emphasizes that $c^{\tilde{I}}_{o''}(\tilde{u}|_{H_1})$ is computed on the submanifold H_1. As

$$\sup\{c^{\tilde{I}}_o(\tilde{u})|\tilde{I} \subseteq \mathcal{O}_o \ \&\ \tilde{I}|_H = I\} = c^I_{o'}(\tilde{u}|_H) = 1,$$

then it follows that

$$\sup\{c^{\tilde{I}}_{o''}(\tilde{u})|\tilde{I} \subseteq \mathcal{O}_{o''} \ \&\ \tilde{I}|_H = I\} = c_{o'}(\tilde{u}|_H) = 1.$$

As $H_1 \cap Y = H \cap Y$, then $\{z|\mathcal{I}(\tilde{u}|_{H_1})_{z''} \neq \mathcal{O}_{z''}, z \in H_1\} \subseteq H \cap Y$. By Corollary 3.3 on H_1 ($u \sim \tilde{u}|_{H_1}$, $J \sim I$), it follows that there exists $N > 0$ (independent of $\tilde{I} \subseteq \mathcal{O}_{o''}$) such that

$$\sup_{\tilde{I}|_H = I} \{c^{\tilde{I}}_{o''}(\max\{\tilde{u}|_{H_1}, N \log |z_{k+1}|\})\} \leq 1. \tag{4.1}$$

By Corollary 3.1, it follows that

$$\sup_{\tilde{I}|_H = I} \{c^{\tilde{I}}_{o''}(\max\{\tilde{u}|_{H_1}, N\log|z_{k+1}|\})\} > 1,$$

which contradicts inequality 4.1.

Then the present theorem has been proved.

5. Proofs of the applications of Theorem 2.1

In this section, we present the proofs of applications of Theorem 2.1.

5.1. **Proof of Theorem 2.2.** As $c^I_{o'}(u|_H) = c$ (equality 1.1), $c_{z'}(u|_H) \leq c$ for any $z' \in (A \cap H)$ (Proposition 1.1) and $\dim((A\cap H, o) \setminus (V(I), o)) = \dim_o(A \cap H)$, by using Remark 3.2 ($o \sim o'$, $u \sim u|_H$, $A \sim A \cap H$), it follows that there exists $z_0 \in ((A \cap H, o') \setminus (V(I), o'))$ such that $c_{z'_0}(u|_H) = c^I_{o'}(u|_H)$ and $\dim_{z_0}(A \cap H) = \dim_o(A \cap H)$.

Note that $\dim_o A \geq \dim_{z_0} A$, then it suffices to consider:

$$\text{"}c_{o'}(u|_H) = c_o(u)\text{"} \Rightarrow \text{"}\dim_o A = n - k + \dim_o(A \cap H)\text{"}$$

($\dim_o A \geq \dim_{z_0} A = n - k + \dim_{z_0}(A \cap H) = n - k + \dim_o(A \cap H)$, $z_0 \sim o$ in the first " $=$ ").

By Remark 3.5 ($u \sim c_o(u)u$, $J = \mathcal{I}(c_o(u)u)_o$), it follows that

$$c_z(\tilde{u}) \leq 1 \tag{5.1}$$

for any $z \in (A, o)$ and $c_o(\tilde{u}) = 1$ ($\Leftarrow c_o(c_o(u)u) = 1$). By Proposition 1.1, it follows that

$$c_{z'}(\tilde{u}|_H) \leq 1 \tag{5.2}$$

for any $z \in (A \cap H, o)$.

Using Proposition 1.1 and inequality 5.2, one can obtain that $c_{o'}(\tilde{u}|_H) \leq c_o(\tilde{u}) \leq 1$. Combining with $1 = c_o(u)/c_o(u) = c_{o'}(u|_H)/c_o(u) = c_{o'}(c_o(u)u|_H) \leq c_{o'}(\tilde{u}|_H)$ ($\Leftarrow c_o(u)u \leq \tilde{u}$), one can obtain that

$$c_{o'}(\tilde{u}|_H) = c_o(\tilde{u}) = 1. \tag{5.3}$$

Let $l = k - \dim_o(A \cap H)$. Let A_3 be a irreducible component of $A \cap H$ on $\mathbb{B}^{k-l} \times \mathbb{B}^l \subset H$ through o satisfying $\dim_o A_3 = k - l$.

By the parametrization of (A_3, o) in H (see "Local parametrization theorem" (4.19) in [10]), it follows that one can find local coordinates $(z_1, \cdots, z_n)$ of a neighborhood $U = \mathbb{B}^{k-l} \times \mathbb{B}^l \times \mathbb{B}^{n-k}$ of o satisfying $H = \{z_{k+1} = \cdots = z_n = 0\}$ and $\dim(A \cap U) = \dim_o A$ such that

(1) $A_3 \cap ((\mathbb{B}^{k-l} \times \mathbb{B}^l) \cap H)$ is reduced and irreducible;

(2) for any $a = (a_1, \cdots, a_{k-l}) \in \mathbb{B}^{k-l}$, $A_3 \cap L_a \neq \emptyset$, where $L_a = \{z_1 = a_1, \cdots, z_{k-l} = a_{k-l}\}$;

(3) there exists analytic subset $A_4 \subseteq \mathbb{B}^{k-l}$ such that any $z \in (A_3 \cap p^{-1}(\mathbb{B}^{k-l} \setminus A_4))$ is the regular point in A_3 and the noncritical point of $p|_{A_{3,reg}}$, where $p : (z_1, \cdots, z_k) = (z_1, \cdots, z_{k-l})$.

By (2), (3), $c_o(\tilde{u}) = c_{o'}(\tilde{u}|_H) = 1$ (inequality 5.3), $c_z(\tilde{u}) \leq 1$ for any $z \in A_3$ (inequality 5.1), and Remark 3.7, it follows that for almost every $a = (a_1, \cdots, a_{k-l}) \in \mathbb{B}^{k-l}$ with respect to the Lebesgue measure on $\mathbb{C}^{k-l}$, there exists $z_a \in A_3 \cap L_a$ such that equality $c_{z'_a}(\tilde{u}|_{L_a}) = 1$ (the set of a denoted by A_{ae}), where z'_a emphasizes that $c_{z'_a}(\tilde{u}|_{L_a})$ is computed on the submanifold L_a.

Let $\tilde{L}_a = \{z_1 = a_1, \cdots, z_{k-l} = a_{k-l}\}$. By inequality 5.1 and Proposition 1.1, it follows that for any $a \in A_{ae}$, $1 = c_{z'_a}(\tilde{u}|_{L_a}) \le c_{z''_a}(\tilde{u}|_{\tilde{L}_a}) \le c_{z_a}(\tilde{u}) = 1$, which implies $c_{z'_a}(\tilde{u}|_{L_a}) = c_{z''_a}(\tilde{u}|_{\tilde{L}_a}) = 1$, where z''_a emphasizes that $c_{z''_a}(\tilde{u}|_{\tilde{L}_a})$ is computed on the submanifold $\tilde{L}_a$.

Using Corollary 2.1 ($\mathbb{C}^n \sim \tilde{L}_a$, $H \sim H \cap \tilde{L}_a = L_a$, $o \sim z_a$, $u \sim \tilde{u}|_{\tilde{L}_a}$), one can obtain that for any $a \in A_{ae}$, $\max_{z_a \in p^{-1}(a)} \dim_{z_a}\{z''|c_{z''}(\tilde{u}|_{\tilde{L}_a}) \le 1\} \ge n-l-(k-l) = n-k$. By the definition of $\tilde{u}$, it follows that $((A \cap U) \cap \tilde{L}_a) \supseteq \{z''|c_{z''}(\tilde{u}|_{\tilde{L}_a}) \le 1\}$, which implies $\dim((A \cap U) \cap \tilde{L}_a) \ge \max_{z_a \in p^{-1}(a)} \dim_{z_a}\{z''|c_{z''}(\tilde{u}|_{\tilde{L}_a}) \le 1\}$. Then we obtain that the $2(n-k)$-dimensional Hausdorff measure of $\dim((A \cap U) \cap \tilde{L}_a)$ is not zero for any $a \in A_{ae}$.

Note that the $2(k-l)$-dimensional Hausdorff measure of A_{ae} is not zero, then it follows that the $2(n-k)+2(k-l) = 2(n-l)$-dimensional Hausdorff measure of A near o is not zero (see Theorem 3.2.22 in [19]), which implies that $\dim_o A = \dim(A \cap U) \ge n-l$. Note that $l = k - \dim_o(A \cap H)$ implies $\dim_o A \le n-k+(k-l) = n-l$, then Theorem 2.2 has been proved.

5.2. **Proof of Theorem 2.3.** By Corollary 3.4, it follows that (2) $\Leftrightarrow$ (3).

In order to prove Theorem 2.3, by Theorem 2.2, it suffices to prove the following statement ((1) $\Rightarrow$ (2)).

Assume that $(A \cap H, o)$ is regular, and $k - \dim_o A \cap H = n - \dim_o A$. If $c_o(u) = c_o(u|_H)$, then there exist coordinates $(w_1, \cdots, w_k, z_{k+1}, \cdots, z_n)$ near o and $l \in \{1, \cdots, k\}$, such that $(w_1 = \cdots = w_l = 0, o) = (A, o)$.

Let $J_0 = \mathcal{I}(c_o(u)u)_o$. By Remark 3.5 ($u \sim c_o(u)u$), it follows that there exists $p_0 > 0$ large enough, such that $\tilde{u} := \max\{c_o(u)u, p_0 \log|J_0|\}$ satisfies: (1) $c_o(\tilde{u}) = 1$ ($\Leftarrow c_o(c_o(u)u) = 1$); (2) $(\{z|c_z(\tilde{u}) \le 1\}, o) = (A, o)$.

By $\tilde{u}|_H \ge c_o(u)u|_H = c_{o'}(u|_H)u|_H$, it follows that $c_{o'}(\tilde{u}|_H) \ge c_{o'}(c_o(u)u|_H) = c_{o'}(c_{o'}(u|_H)u|_H) = 1$. Combining with the fact that $c_{o'}(\tilde{u}|_H) \le c_o(\tilde{u}) = 1$, we obtain that

$$c_{o'}(\tilde{u}|_H) = 1. \tag{5.4}$$

Note that $c_{z'}(\tilde{u}|_H) \le c_z(\tilde{u})$ for any $z \in A \cap H$, then by (2) ($\Rightarrow c_z(\tilde{u}) \le 1$) for any $z \in A \cap H$, it follows that $(\{z|c_{z'}(\tilde{u}|_H) \le 1\}, o) \supseteq (A \cap H, o)$. Combining with the definition of $\tilde{u}$ ($\Rightarrow (\{z|c_{z'}(\tilde{u}|_H) < +\infty\}, o) \subseteq (V(J_o) \cap H, o) = (A \cap H, o)$), we obtain

$$(V(\mathcal{I}(\tilde{u}|_H)), o) = (\{z|c_{z'}(\tilde{u}|_H) \le 1\}, o) = (A \cap H, o). \tag{5.5}$$

In the following part of the present section, we consider $\tilde{u}$ instead of u.

By equality 5.5, it follows that $(V(\mathcal{I}(\tilde{u}|_H)), o)(= (A \cap H, o))$ is regular. Combining with equality 5.4 and Corollary 3.4 ($u \sim \tilde{u}|_H$), it follows that there exist $l \in \{1, \cdots, k\}$ and holomorphic functions $f_1, \cdots, f_l$ near $o' \in H$ such that
(a) $df_1|_{o'}, \cdots, df_l|_{o'}$ are linear independent;
(b) $(\{f_1 = \cdots = f_l = 0\}, o) = (A \cap H, o)$ holds;
(c) $|f_j|^2 e^{-2\tilde{u}|_H}$ are all locally integrable near o' for $j \in \{1, \cdots, l\}$.

By Remark 3.1 and (c), it follows that there exist holomorphic functions $F_1, \cdots, F_l$ near $o \in \mathbb{C}^n$ such that and $|F_j|^2 e^{-2\tilde{u}}$ are integrable near o for any $j \in \{1, \cdots, l\}$, which implies that $\{F_1 = \cdots = F_l = 0\} \supseteq A$. Combining $F_j = f_j$ and (a), we obtain that $dF_1|_o, \cdots, dF_l|_o, dz_{k+1}|_o, \cdots, dz_n|_o$ are linear independent.

Note that $\{F_1 = \cdots = F_l = 0\}$ is regular near o and $n - \dim_o A = k - \dim_o A \cap H = l$, then it follows that $\{F_1 = \cdots = F_l = 0\} = A$ near o. Choosing $w_j = F_j$ for any $j \in \{1, \cdots, l\}$, one can find holomorphic functions $w_{l+1}, \cdots, w_k$ near o such that $dw_1|_o, \cdots, dw_k|_o, dz_{k+1}|_o, \cdots, dz_n|_o$ are linear independent. Then Theorem 2.3 has been proved.

5.3. **Proof of Remark 2.2.** Let $A_1 = V(\mathcal{I}(cu))$ and $A_2 = V(\mathcal{I}(cv))$, and $A = \{(z,w)|c_{(z,w)}(\max\{\pi_1^*(u), \pi_2^*(v)\}) \leq c\}$. By Proposition 1.2, it follows that

$$\begin{aligned} c_{(o,o)}(\max\{\pi_1^*(u), \pi_2^*(v)\}) &= c_o(u) + c_o(v) = c \\ &= c_o(\max\{u, v\}) = c_{(z,w)}(\max\{\pi_1^*(u), \pi_2^*(v)\}|_\Delta). \end{aligned}$$

Using Theorem 2.2 ($u \sim \max\{\pi_1^*(u), \pi_2^*(v)\}$, $H \sim \Delta$, $o \sim (o,o)$, $k \sim n$, $n \sim 2n$), we obtain $\dim_{(o,o)} A = \dim_{(o,o)}(A \cap \Delta) + n$. By Proposition 1.2, it follows that $A = \{(z,w)|c_z(u) + c_w(v) \leq c\} \subseteq \{(z,w)|\max\{c_z(u), c_w(v)\} \leq c\} = A_1 \times A_2$, which implies $\dim_o A_1 + \dim_o A_2 = \dim_{(o,o)}(A_1 \times A_2) \geq \dim_{(o,o)} A$. Note that $B = \{z|c_z(u) + c_z(v) \leq c\}$ is biholomophic to $A \cap \Delta$, then it follows that $\dim_o A_1 + \dim_o A_2 \geq \dim_{(o,o)} A = \dim_{(o,o)}(A \cap \Delta) + n = n + \dim_o B$. Remark 2.2 has thus been proved.

5.4. **Proof of Proposition 2.1.** Following the symbols in subsection 5.3, by Theorem 2.3 ($n \sim 2n$, $k \sim n$, $u \sim \max\{\pi_1^* u, \pi_2^* v\}$, $o \sim (o,o) \in \mathbb{C}^n \times \mathbb{C}^n$, $H \sim \Delta$ the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$), it follows that A is regular at $((o,o))$ satisfying $\dim_{(o,o)} A = \dim_{(o,o)}(A \cap \Delta) + n$. As $A_1 \cap A_2 = B$, it follows that $(A_1 \times A_2) \cap \Delta = A \cap \Delta$, which implies

$$\dim_{(o,o)} A = \dim_{(o,o)}(A \cap \Delta) + n = \dim_{(o,o)}((A_1 \times A_2) \cap \Delta) + n. \tag{5.6}$$

Note that $A_1 \times A_2 \supseteq A$ and equality 5.6 holds, then it follows that $\dim_{(o,o)}(A_1 \times A_2) \geq \dim_{(o,o)} A = \dim_{(o,o)}((A_1 \times A_2) \cap \Delta) + n$. As Δ is regular, then it is clear that $\dim_{(o,o)}(A_1 \times A_2) \leq \dim_{(o,o)}((A_1 \times A_2) \cap \Delta) + n$, which implies $\dim_{(o,o)}(A_1 \times A_2) = \dim_{(o,o)}((A_1 \times A_2) \cap \Delta) + n = \dim_{(o,o)} A$. Note that $(A, (o,o))$ is regular and $A_1 \times A_2$ is irreducible at (o,o) (A_1 and A_2 are both irreducible at o), then we obtain $A = A_1 \times A_2$, which implies A_1 and A_2 are both regular.

By the transversality between $A_1 \times A_2 = A$ and Δ at (o,o) and Remark 3.6, it follows that $2n = \dim(T_{A_1 \times A_2,(o,o)} + T_{\Delta,(o,o)}) = \dim T_{A_1 \times A_2,(o,o)} + \dim T_{\Delta,(o,o)} - \dim(T_{A_1 \times A_2,(o,o)} \cap T_{\Delta,(o,o)}) = (\dim T_{A_1,o} + \dim T_{A_2,o}) + n - \dim(T_{A_1,o} \cap T_{A_2,o}) = \dim(T_{A_1,o} + T_{A_2,o}) + n$. It is clear that $\dim(T_{A_1,o} + T_{A_2,o}) = n$, then we prove Proposition 2.1.

6. Proofs of two sharp relations on jumping numbers

In the present section, we prove Theorem 2.5 and Theorem 2.6.

6.1. **Proof of Theorem 2.5.**

By $c_o^I(\tilde{u}_l) = 1$ in Remark 3.5, it follows that $c_o^I(\max\{c_o^I(u)u, \frac{1}{\frac{c_o^{IJ}(u)}{c_o^I(u)} - 1} \log |J|\}) = 1$.

By the monotonicity of complex singularity exponents ($u \le v \Rightarrow c_o(u) \le c_o(v)$), it follows that $c_o^I(\frac{1}{\frac{c_o^{IJ}(u)}{c_o^I(u)}-1}\log|J|\}) \le 1$, i.e.,

$$\frac{c_o^I(u)}{c_o^{IJ}(u)-c_o^I(u)} \ge c_o^I(\log|J|). \tag{6.1}$$

Then Theorem 2.5 has thus been proved.

For the sake of completeness, we give a proof of the following equivalence

$$IJ \subseteq \mathcal{I}(c_o^I(u)u)_o \Leftrightarrow c_o^{IJ}(u) > c_o^I(u).$$

Firstly, we prove "$\Rightarrow$". Since $IJ \subseteq \mathcal{I}(c_o^I(u)u)_o$ implies that $|IJ|^2e^{-2c_o^I(u)u}$ is locally integrable near o, then it follows that $c_o^{IJ}(u) > c_o^I(u)$ by the strong openness property.

Secondly, we prove "$\Leftarrow$". Since $IJ \not\subseteq \mathcal{I}(c_o^I(u)u)_o$ implies that $|IJ|^2e^{-2c_o^I(u)u}$ is not locally integrable near o, then it follows that $c_o^{IJ}(u) \le c_o^I(u)$ by the definition of $c_o^{IJ}(u)$.

6.2. **Proof of Corollary 2.2.**

When $I = \mathcal{O}_o$, inequality 6.1 degenerates to

$$\frac{c_o(u)}{c_o^J(u)-c_o(u)} \ge c_o(\log|J|), \tag{6.2}$$

i.e.,

$$c_o^J(u) \le \frac{c_o(u)}{c_o(\log|J|)} + c_o(u). \tag{6.3}$$

Then Corollary 2.2 follows for $J \subseteq \mathcal{I}(c_o(u)u)_o$.

If J does not satisfy $J \subseteq \mathcal{I}(c_o(u)u)_o$, then $|J|^2e^{-2c_o(u)u}$ is not integrable near o, which implies $c_o^J(u) \le c_o(u)$. Note that $|J|^2$ is locally bounded near o. Then it follows that $|J|^2e^{-2cu}$ is locally integrable near o for any $c < c_o(u)$, which implies $c_o^J(u) \ge c_o(u)$. Then it is clear that $c_o^J(u) = c_o(u)$.

Then Corollary 2.2 has been proved.

6.3. **Proof of Theorem 2.6.**

By Corollary 3.2, it follows that for any $\varepsilon > 0$, there exists a coherent ideal $\tilde{I}|_H = I$ such that

$$c_o^{\tilde{I}}(\max(\varphi, \frac{N-1}{b}\log|h|)) \ge \frac{bN}{N-1} - \varepsilon. \tag{6.4}$$

Assume that $b_1 > 0$. For any $\tilde{I}|_H = I$, it follows that

$$\frac{1}{c_o^{\tilde{I}h}(\varphi) - c_o^{\tilde{I}}(\varphi)} \le \frac{1}{b_1}. \tag{6.5}$$

By Remark 3.5 ($l \sim \frac{c_o^{\tilde{I}h}(\varphi)}{c_o^{\tilde{I}}(\varphi)}$, $I \sim \tilde{I}$, $J \sim h$), it follows that

$$c_o^{\tilde{I}}(\max\{c_o^{\tilde{I}}(\varphi)\varphi, \frac{1}{\frac{c_o^{\tilde{I}h}(\varphi)}{c_o^{\tilde{I}}(\varphi)}-1}\log|h|\}) = 1. \tag{6.6}$$

By Corollary 3.2 ($k \sim n-1$, $z_{k+1} \sim h$, $b \sim c_{o'}^I(\varphi|_H)$, $\frac{N-1}{b} \sim \frac{1}{b_1}$),

which deduces

$$\sup_{\tilde{I}|_H=I} c_o^{\tilde{I}}(\max\{\varphi, \frac{1}{b_1}\log|h|\}) \geq \frac{c_{o'}^I(\varphi|_H)(c_{o'}^I(\varphi|_H)+b_1)}{c_{o'}^I(\varphi|_H)} = c_{o'}^I(\varphi|_H)+b_1 \qquad (6.7)$$

By equality 6.6, it follows that

$$c_{o'}^I(\varphi|_H)+b_1 \leq b_0$$

holds for any $\varepsilon > 0$ (if not, then " $>$ " holds. Combining with inequality 6.5, it follows that there exists $\tilde{I}$ such that

$$c_o^{\tilde{I}}(\max\{\varphi, (\frac{1}{c_o^{\tilde{I}h}(\varphi)-c_o^{\tilde{I}}(\varphi)})\log|h|\}) \geq c_o^{\tilde{I}}(\max\{\varphi, \frac{1}{b_1}\log|h|\}) > b_0 \geq c_o^{\tilde{I}}(\varphi) \quad (6.8)$$

that contradicts equality 6.6). Then Theorem 2.6 is proved for the case $b_1 > 0$.

When $b_1 \leq 0$, noting that $c_{o'}^I(\varphi|_H) \leq b_0$ (Restriction formula (jumping number)), we prove Theorem 2.6.

6.4. **Proof of Corollary 2.3.**

By Remark 6.1, it suffices to consider $k = n-1$.

Consider the holomorphic map $p : \mathbb{C}^n \to \mathbb{C}^n$ with coordinates $(z_1, \cdots, z_n)$ and $(w_1, \cdots, w_n)$ respectively satisfying $p(z_1, \cdots, z_{n-1}, z_n) = (z_1, \cdots, z_{n-1}, z_{n-1}z_n)$. Then it follows that

$$\int_{\Delta_r^n} e^{-2l\varphi} = \int_{\Delta_r(a)}\int_{\Delta_r^n\cap\{z_n=a\}} |z_{n-1}|^2 e^{-2l\varphi\circ p},$$

which implies $c_{o'}^{w_{n-1}}(\varphi|_{\frac{w_n}{w_{n-1}}=a}) = c_{(0,\cdots,0,a)'}^{z_{n-1}}((\varphi\circ p)|_{z_n=a}) \geq c_n$ for a.e. $a \in \Delta_r$ ($r > 0$ small enough, using lower semicontinuity of complex singularity exponent), where $(0,\cdots,0,a)'$ emphasizes that $c_{(0,\cdots,0,a)'}^{z_{n-1}}((\varphi\circ p)|_{\{z_n=a\}})$ is computed on the submanifold $\{z_n = a\}$.

Using inequality 2.2 ($n \sim n-1$, $h(= z_n) \sim w_{n-1}$, $\varphi \sim \varphi|_{\frac{w_n}{w_{n-1}}=a}$, $c_o^h(\varphi) \sim c_{o'}^{w_{n-1}}(\varphi|_{\frac{w_n}{w_{n-1}}=a})$), we obtain Corollary 2.3.

Remark 6.1. *For any $k \in \{1, \cdots, n-2\}$, there exist dim k and $k+2$ planes H_k and H_{k+2} planes through o satisfying $H_k \subset H_{k+2} \subset \mathbb{C}^n$, such that $c_{o'}(\varphi|_{H_k}) = c_k$ and $c_{o''}(\varphi|_{H_{k+2}}) = c_{k+2}$, where o'' emphasizes that $c_{o''}(\varphi|_{H_{k+2}})$ is computed on the submanifold H_{k+2}.*

By Remark 2.5, it suffices to consider the following remark (proof see Section 6.5).

Remark 6.2. *Let G_1 and G_2 be two subsets of $G(k_1, n)$ and $G(k_2, n)$ whose complements are of $U(n)$-invariant measure 0 respectively ($k_1 < k_2$). Then there exists $V_1 \in G_1$ and $V_2 \in G_2$ satisfying $V_1 \subset V_2$.*

6.5. **Proof of Remark 6.2.**

It is well-known that a Zariski open set of $G(k_1, n)$ ($G(k_2, n)$) could be presented as $(M(k_1, n-k_1)\,(\delta_{j,k_1+1-l})_{1\leq j,l\leq k_1})$ $((M(k_2, n-k_2)\,(\delta_{j,k_2+1-l})_{1\leq j,l\leq k_2}))$ with respect to the same coordinate $(z_1, \cdots, z_n)$.

We consider two cases: (1) $n \geq k_1 + k_2$; (2) $n < k_1 + k_2$.

proof of Case (1)

Let $D \in M(k_2 - k_1, n - (k_1 + k_2))$. Consider a family of mappings p_D from $M(k_1, n - k_1)$ to $(M(k_2, n - k_2), (\delta_{j,k_2+1-l})_{1\le j\le k_2, 1\le l\le k_2-k_1})$:

$$p_D(A\,B) := \begin{pmatrix} A - B \times (\delta_{j,k_2-k_1+1-l})_{1\le j,l\le k_2-k_1} \times (D\,B^t) & 0 \\ (D\,B^t) & (\delta_{j,k_2-k_1+1-l})_{1\le j,l\le k_2-k_1} \end{pmatrix} \tag{6.9}$$

i.e.,

$$\begin{pmatrix} (\delta_{j,l})_{1\le j,l\le k_1} & -B \times (\delta_{j,k_2-k_1+1-l})_{1\le j,l\le k_2-k_1} \\ 0 & (\delta_{j,l})_{1\le j,l\le k_2-k_1} \end{pmatrix} \begin{pmatrix} A & B \\ (D, B^t) & (\delta_{j,k_2-k_1+1-l})_{1\le j,l\le k_2-k_1} \end{pmatrix}, \tag{6.10}$$

for any $A \in M(k_1, n - k_2)$ and $B \in M(k_1, k_2 - k_1)$.

Note that

(1a) for any D, holomorphic map p_D is injective;

(1b) $\sqcup_D p_D(M(k_1, n - k_1)) = (M(k_2, n - k_2)\,(\delta_{j,k_2+1-l})_{1\le j\le k_2, 1\le l\le k_2-k_1})$, which implies that for a.e. $D \in M(k_2 - k_1, n - (k_1 + k_2))$ and a.e. $M \in M(k_1, n - k_1)$ $(P_D(M)\,(\delta_{j,k_1+1-l})_{1\le j,l\le k_1}) \in G_2$.

(1c) the vector space generated by the row vector of $(p_D(A\,B)\,(\delta_{j,k_1+1-l})_{1\le j,l\le k_1})$ contains the vector space generated by the row vector of $(A\,B\,(\delta_{j,k_1+1-l})_{1\le j\le k_2, 1\le l\le k_1})$ (by equality 6.10).

By (1b) and (1c), it follows that case (1) has been proved.

Proof of Case (2)

Let $D \in M(k_2 - k_1, k_1 + k_2 - n)$. Consider a family of mappings p_D from subset $G_D := \{A\,(D\,B^t)^t | A \in M(k_1, n - k_2), B \in M(n - k_2, k_2 - k_1)\}$ of $M(k_2, n - k_2)$ to $(M(k_1, n - k_1)\,(\delta_{j,k_2+1-l})_{1\le j\le k_2, 1\le l\le k_2-k_1})$:

$$p_D(A\,(D\,B^t)^t) := \begin{pmatrix} A - (D\,B^t)^t \times (\delta_{j,k_2-k_1+1-l})_{1\le j,l\le k_2-k_1} \times B^t & 0 \\ B^t & (\delta_{j,k_2-k_1+1-l})_{1\le j,l\le k_2-k_1} \end{pmatrix} \tag{6.11}$$

i.e.,

$$\begin{pmatrix} (\delta_{j,l})_{1\le j,l\le k_1} & -(D\,B^t)^t \times (\delta_{j,k_2-k_1+1-l})_{1\le j,l\le k_2-k_1} \\ 0 & (\delta_{j,l})_{k_1+1\le j,l\le k_2} \end{pmatrix} \begin{pmatrix} A & (D\,B^t)^t \\ B^t & (\delta_{j,k_2-k_1+1-l})_{1\le j,l\le k_2-k_1} \end{pmatrix}, \tag{6.12}$$

for any $A \in M(k_1, n - k_2)$ and $B \in M(n - k_2, k_2 - k_1)$.

Note that

(2a) for any D, holomorphic map p_D is surjective and injective;

(2b) $\sqcup_D G_D = (M(k_2, n-k_2)\,(\delta_{j,k_2+1-l})_{1\le j\le k_2, 1\le l\le k_2-k_1})$, which implies that for a.e. $D \in M(k_2 - k_1, k_1 + k_2 - n)$ and a.e. $M \in G_D$, $(P_D(M)\,(\delta_{j,n-l})_{1\le j,l\le k_1}) \in G_1$.

(2c) the vector space generated by the row vector of $(p_D(A\,(D\,B^t)^t)\,(\delta_{j,k_1+1-l})_{1\le j\le k_2, 1\le l\le k_1})$ contains the vector space generated by the row vector of $(A\,(D\,B^t)^t\,(\delta_{j,k_1+1-l})_{1\le j,l\le k_1})$ (by equality 6.12).

By (2b) and (2c), it follows that case (2) has been proved.

7. Berndtsson's log subharmonicity and integrability

In this section, we present a relationship between Berndtsson's log subharmonicity and integrability.

We recall a lemma which was used in [24, 25, 26] to prove Demailly's strong openness conjecture:

Lemma 7.1. *(see [24, 25]) Let h_a be a holomorphic function on unit disc $\Delta \subset \mathbb{C}$ which satisfies $h_a(o) = 0$ and $h_a(a) = 1$ for any a, then we have*

$$\int_{\Delta_r} |h_a|^2 d\lambda_1 > C_1 |a|^{-2},$$

where $a \in \Delta$ whose norm is smaller than $\frac{1}{6}$, C_1 is a positive constant independent of a and h_a.

Let u be a plurisubharmonic function on $\Delta^n \times \Delta^m$ $(n = k, m = 1)$ with coordinates $(z_1, \cdots, z_k, w)$, and p be the projection with $p(z_1, \cdots, z_k, w) = w$ and K_{2u} be the fiberwise Bergman kernel as in the above subsection.

Proposition 7.1. *If $u > 0$, then e^{-2u} is integrable near the origin $(o, o_w) \in \mathbb{C}^{k+1}$ if and only if $K_{2u}^{-1}(o, w)$ is integrable near the origin o_w with respect to w, i.e.,*

$$\nu(\frac{1}{2} \log K_{2u}(o, \cdot), o_w) \geq 1.$$

Proof. It is clear that if e^{-2u} is integrable near origin (o, o_w), then $K_{2u}^{-1}(o, w)$ is integrable near o_w. Then it suffices to prove "only if" part, i.e., if e^{-2u} is not integrable near (o, o_w), then $K_{2u}^{-1}(o, w)$ is not integrable near o_w.

We use our idea of movably using Ohsawa-Takegoshi L^2 extension theorem ([24, 25, 26]) to prove "only if" part:

As e^{-2u} is not integrable near o, then it follows from Theorem 3.1 $(H = p^{-1}(a))$ that for any $a \in \Delta$, there exists holomorphic function F_a on Δ^{k+1} such that

(1) $F_a(o, a) = 1$;

(2) $\int_{\Delta^{k+1}} |F_a|^{-2u} \leq C_D K_{2u}^{-1}(o, a)$;

(3) $F_a(o, o_w) = 0$.

(Using the definition of K_{2u}, one can choose holomorphic f_a on $\Delta^k \times \Delta$ satisfying $f_a(o, a) = 1$ and $\int_{p^{-1}(a)} |f_a|^2 e^{-2u} = K_{2u}^{-1}(o, a)$, and F_a is the Ohsawa-Takegoshi L^2 extension of f_a.)

By Lemma 7.1 $(F_a(z_1, \cdots, z_k, \cdot) = h_a(\cdot))$ and the submean inequality of $|F_a|^2$, it follows that $\int_{\Delta^{k+1}} |F_a|^2 > C_2 \frac{1}{|a|^2}$, where $C_2 > 0$ is independent of a. As $u > 0$, then it follows from (2) that $K_{2u}^{-1}(o, a) \geq \frac{1}{C_D} \int_{\Delta^{k+1}} |F_a|^2 e^{-2u} > \frac{C_2}{C_D} \frac{1}{|a|^2}$. Then the present Proposition has been done. □

Remark 7.1. *If $e^{-2u|_{p^{-1}(0)}}$ is integrable near o, then $e^{-2u-2c\log|w|}$ is also integrable near (o, o_w), where $c \in (0, 1)$.*

Proof. As $e^{-2u|_{p^{-1}(0)}}$ is integrable near o, it follows that $\nu(\log K_{2u}(o, \cdot), o_w) = 0$. Since $K_{2u+2c\log|\cdot|}(o, \cdot) = |\cdot|^{2c} K_{2u}(o, \cdot)$, then $\nu(\frac{1}{2} \log K_{2u+2c\log|\cdot|}(o, \cdot), o_w) = c < 1$. By Proposition 7.1, the present Remark has thus been done. □

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Qi'an Guan: School of Mathematical Sciences, and Beijing International Center for Mathematical Research, Peking University, Beijing, 100871, China.

E-mail address: guanqian@amss.ac.cn

Xiangyu Zhou: Institute of Mathematics, AMSS, and Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing, China

E-mail address: xyzhou@math.ac.cn